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VOL. 272 NO. 4

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ON NONLINEAR OSCILLATIONS IN ELECTROMECHANICAL SYSTEMS *

BY

Y. H. KU [1]

ABSTRACT

Part I of the paper gives the representation of an electromechanical system by the dynamic equation of Lagrange in the holonomic reference frame. The equation of voltage and the equation of torque are then obtained from Lagrange's equation.

Part II is concerned with the transformation of the system from the holonomic frame to the nonholonomic frame. Lagrange's equation is generalized to the Boltzmann-Hamel equation. Theory of hunting is presented by using a uniformly rotating reference frame.

Part III deals with the transformation of the system from the holonomic frame to the quasi-holonomic reference frame in which the d, q components (Park variables) and the f, b components (Ku variables) are developed. From the equation of voltage, the impedance matrix is obtained by considering ω, the angular velocity, as a constant. The equation of torque is, however, nonlinear. If ω is considered as the derivative of the angular displacement θ, the "fourth dimension" in an electromechanical system, the system of equations are *nonlinear*. A simple four-dimensional formulation thus represents the dynamic characteristic of an electromechanical system. Four simultaneous first-order nonlinear differential equations are explicitly given.

Equations of hunting in the nonholonomic frame and equations of voltage for transient analysis in the quasi-holonomic frame, are given in Appendices I and II, respectively.

* Presented at the International Symposium on Nonlinear Vibrations, Kiev, USSR, September 12–18, 1961. This symposium was organized under the auspices of the International Union of Theoretical and Applied Mechanics, of which the author has been an elected member of the General Assembly since 1946.

[1] The Moore School of Electrical Engineering, University of Pennsylvania, Philadelphia, Pa.

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Part I

1. THE EQUATION OF LAGRANGE

An electromechanical system can be represented by the dynamic equation of Lagrange:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^\gamma}\right) - \frac{\partial T}{\partial x^\gamma} + \frac{\partial F}{\partial \dot{x}^\gamma} = e_\gamma \tag{1}$$

where T denotes the *total* kinetic energy function and F denotes the *total* energy loss in the electromechanical system, x^γ denotes the coordinates, $\dot{x}^\gamma$ their derivatives with respect to time, and e_γ denotes the electromotive forces or torques. The total kinetic energy function is given by the quadratic form

$$T = \tfrac{1}{2} a_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \tag{2}$$

which comprises both the energy stored in the magnetic field W_m and the energy stored in a rotating body W_ω, where

$$W_m = T_m = \tfrac{1}{2} L_{mn} i^m i^n \tag{3}$$

$$W_\omega = T_\omega = \tfrac{1}{2} J_{uv} \omega^u \omega^v. \tag{4}$$

The total energy loss (the Rayleigh dissipation function) in the electromechanical system comprises both the electrical loss F_e and the friction loss in the rotating mechanical system F_f. In quadratic form these are:

$$F = \tfrac{1}{2} r_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \tag{5}$$

$$F_e = \tfrac{1}{2} W_e = \tfrac{1}{2} R_{mn} i^m i^n \tag{6}$$

$$F_f = \tfrac{1}{2} W_f = \tfrac{1}{2} B_{uv} \omega^u \omega^v. \tag{7}$$

We see that the symbol $\dot{x}^\alpha$, denoting the derivative of x^α with respect to time, represents either the electric current i^m or the angular velocity ω^u. Hence the symbol x^α, denoting the coordinate, represents either the electric charge q^m or the angular displacement θ^u. Note that x^γ, where the index γ includes all values of α and β, is a contravariant tensor, and e_γ is a covariant tensor. The indices (α, β) in Eqs. 2 and 5 include the electrical coordinates (m, n) and the mechanical coordinates (u, v). Since we have introduced two sets of indices (m, n) and (u, v), there is no need to use the symbols q, θ, i, ω, L, J, R, and B. Thus

$$\left.\begin{aligned} a_{mn} &= L_{mn} \\ a_{uv} &= J_{uv} \\ r_{mn} &= R_{mn} \\ r_{uv} &= B_{uv} \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} \dot{x}^m &= i^m \\ \dot{x}^u &= \omega^u \\ x^m &= q^m \\ x^u &= \theta^u \end{aligned}\right\} \tag{9}$$

Denoting $\dot{x}^\gamma$ by i^γ, Eq. 1 can be rewritten as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial i^\gamma}\right) - \frac{\partial T}{\partial x^\gamma} + \frac{\partial F}{\partial i^\gamma} = e_\gamma. \tag{10}$$

We note that e_m denotes the electromotive forces and e_u denotes the torques.

The coordinates, whether they are electric charges $x^m(q^m)$ or mechanical angles $x^u(\theta^m)$, are rigidly attached to the conductors or the frame. This reference frame is termed *holonomic*, and Lagrange's equation is valid in a holonomic reference frame.

In a rotating machine (electric generator or motor), there are conductors on the stator as well as on the rotor. So i^m would include both stator and rotor currents, and e_m would include both stator and rotor voltages. We further note that the inductance $a_{mn} = L_{mn}$ would include self and mutual inductances. If the effect of saturation is taken into account, the inductance would be a function of the current i^m. The mutual inductance between the stator and the rotor is a function of the angle x^u or θ^u. In a salient-pole machine, there is also second-harmonic variation in the mutual inductance between the stator phases. Thus $a_{mn} = L_{mn}$ is a function of the coordinate x^u.

Substituting Eqs. 2 and 5 into Eq. 1 or 10 gives

$$r_{\gamma\alpha}i^\alpha + a_{\gamma\alpha}\frac{di^\alpha}{dt} + \left(\frac{\partial a_{\gamma\alpha}}{\partial x^\beta} - \frac{1}{2}\frac{\partial a_{\alpha\beta}}{\partial x^\gamma}\right) i^\alpha i^\beta + \frac{d}{dt}\left[\frac{1}{2}\, i^\alpha i^\beta \frac{\partial a_{\alpha\beta}}{\partial i^\gamma}\right] + \frac{\partial a_{\gamma\alpha}}{\partial i^\beta}\left(\frac{di^\beta}{dt}\right) i^\alpha = e_\gamma. \tag{11}$$

Now if $a_{\alpha\beta}$ is not a function of i^α (neglecting saturation), the fourth and fifth terms are dropped and Eq. 11 simplifies to

$$r_{\gamma\alpha}i^\alpha + a_{\gamma\alpha}\frac{di^\alpha}{dt} + [\alpha\beta, \gamma]i^\alpha i^\beta = e_\gamma \tag{12}$$

where the Christoffel symbol is defined by

$$[\alpha\beta, \gamma] = \frac{1}{2}\left(\frac{\partial a_{\gamma\beta}}{\partial x^\alpha} + \frac{\partial a_{\gamma\alpha}}{\partial x^\beta} - \frac{\partial a_{\alpha\beta}}{\partial x^\gamma}\right). \tag{13}$$

Equation 12 was first given by Kron (1)[2] for the electromechanical system and was derived in Bewley's recent book (2).

2. THE EQUATION OF VOLTAGE

If we limit e_γ to e_m, which denotes the electromotive forces or voltages, and assume that a_{mn} is a function of x^u (or θ) only, the equation of voltage is given by

$$r_{mn}i^n + a_{mn}\frac{di^n}{dt} + \frac{\partial a_{mn}}{\partial x^u}i^u i^n = e_m. \tag{14}$$

Although in general x^u would denote $\theta_1, \theta_2, \cdots$, in practice we may need only to consider one angular displacement θ and its corresponding angular velocity ω. Thus Eq. 14 simplifies to

$$e_m = r_{mn}i^n + a_{mn}\frac{di^n}{dt} + \frac{\partial a_{mn}}{\partial \theta}\omega i^n. \tag{15}$$

In order to get a complete solution for the performance of the electromechanical system, the equation of voltage has to be solved simultaneously with the equation of torque. In other words, the complete system represented by Eq. 11 or 12 has to be analyzed as a whole. Even in the non-saturation case, Eq. 15 is *nonlinear* in the sense that the coefficient for i^n in the third term of the right-hand member involves another dependent variable $i^u = \dot{\theta} = \omega$. Unless we *assume* that ω is a *constant* we cannot solve for a set of currents which satisfies Eq. 15. In the transient analysis of rotating machinery, we often do assume that during the short interval of electrical oscillations, ω remains constant, and hence Eq. 15 reduces to a system of varying-parameter equations. However, it is well known that the system cannot be analyzed in a holonomic reference frame. In a synchronous machine study, the stator or armature currents (and voltages) have to be referred to the rotating reference frame (3, 4, 5). In the transient study of induction motors, one can choose to refer the stator quantities to the rotor or the rotor quantities to the stator.

Maxwell's equation of voltage (Faraday's law) as applied to a holonomic reference frame is given as

$$e_m = R_{mn}i^n + \frac{d\phi_m}{dt} = R_{mn}i^n + \frac{d}{dt}(L_{mn}i^n) \tag{16}$$

where $\phi = L_{mn}i^n$ denotes the flux-linkages. Let L_{mn} be both a function of the current i^n and the mechanical angle $i^u = \theta^u$. Then

$$\frac{d}{dt}(L_{mn}i^n) = L_{mn}\frac{di^n}{dt} + i^n\frac{\partial L_{mn}}{\partial i^m}\left(\frac{di^m}{dt}\right) + i^n\frac{\partial L_{mn}}{\partial \theta^u}\omega^u. \tag{17}$$

[2] The boldface numbers in parentheses refer to the references appended to this paper.

Substituting Eq. 17 into Eq. 16 gives

$$e_m = R_{mn}i^n + L_{mn}\frac{di^n}{dt} + \frac{\partial L_{mn}}{\partial \theta^u}\omega^u i^n + \frac{\partial L_{mn}}{\partial i^m}\left(\frac{di^m}{dt}\right) i^n. \tag{18}$$

This expression reduces to Eq. 15 for neglecting saturation and limiting θ^u to one angular displacement θ. The expression $(\partial L_{mn}/\partial\theta)$ is sometimes termed the *motion matrix*.

3. THE EQUATION OF TORQUE

If we limit e_γ to e_u (or τ), which denotes the torque, and assume that a_{mn} is a function of x^u (or θ) only, the equation of torque is given by

$$r_{uv}i^v + a_{uv}\frac{di^v}{dt} - \frac{1}{2}\frac{\partial a_{mn}}{\partial x^u} i^m i^n = e_u = \tau_u. \tag{19}$$

Since we are considering only one machine, there is only one torque and Eq. 19 simplifies to

$$B\omega + J\frac{d\omega}{dt} - \frac{1}{2}\frac{\partial L_{mn}}{\partial \theta^u} i^m i^n = \tau. \tag{20}$$

The superscript of θ^u can be dropped if there is only one angular displacement involved.

The power delivered to a motor or developed from a generator is given by the general expression

$$P = e_m i^m. \tag{21}$$

Substituting Eq. 15 into Eq. 21 gives

$$P = e_m i^m = r_{mn}i^n i^m + a_{mn}\frac{di^n}{dt} i^m + \left(\omega\frac{\partial a_{mn}}{\partial\theta}\right) i^n i^m. \tag{22}$$

On the righthand side of Eq. 22, the first term denotes the ohmic loss, the second term represents the power associated with the inductance voltage drop $a_{mn}(di^n/dt)$, and the third term denotes the "rotational power" which is associated with (ω). In the case of motor, the rotational power represents the useful output, while $P = e_m i^m$ represents the input. Dividing rotational power by ω gives

$$\tau = \frac{\partial a_{mn}}{\partial\theta} i^n i^m. \tag{23}$$

Since $\phi_m = a_{mn}i^n$ according to Eq. 16, Eq. 23 can be rewritten as

$$\tau = \frac{\partial \phi_m}{\partial\theta} i^m. \tag{24}$$

This torque expression was derived by the author (**5**, **6**) for a three-phase synchronous machine as

$$\tau = i^A \frac{\partial \phi_A}{\partial \theta} + i^B \frac{\partial \phi_B}{\partial \theta} + i^C \frac{\partial \phi_C}{\partial \theta}, \tag{25}$$

where i^A, i^B, and i^C denote the phase currents of the armature (stator), and ϕ_A, ϕ_B, and ϕ_C denote the phase flux-linkages. As we have noted, all the quantities in Eqs. 23, 24, and 25 refer to the holonomic reference frame. These expressions agree in form with the third term on the left side of Eq. 19 or 20. As a_{mn} is a function of θ in the case of a general rotating machine with salient-poles, it is easily seen that the expression for the electromagnetic torque given in Eq. 19 gives the same result as in Eq. 25 with a proper sign change.

Part II

4. THE NONHOLONOMIC FRAME

From both the theoretical and the practical points of view one should be able to transform from a *holonomic* system to a *nonholonomic* system in which the reference axes are no longer fixed to the conductors (on the stator and on the rotor) or the frame. A nonholonomic reference frame may be termed a *free* frame. We have mentioned in Section 2 that in the transient study of induction machines, one can choose to refer the stator quantities to the rotor or vice versa. However, if one is interested in connecting an induction machine with a synchronous machine, the rotor of the induction machine will not run at the synchronous speed which is the speed of the synchronous machine. As shown in (**3**, **4**, **5**, **6**), both the stator and rotor quantities have to be referred to a synchronously rotating reference frame before interconnection with a synchronous machine can be made. Physically, the armature terminals of a synchronous generator and the stator terminals of an induction motor can be easily connected together through a constant-parameter network or a distributed-parameter transmission line. In fact one can make the connection in the stationary frame and transform these stator (or stationary frame) quantities to a synchronous rotating reference frame. Taking the induction machine as a simple electromechanical system we can see that it is desirable to refer both the stator quantities and the rotor quantities to a synchronous rotating reference frame. This synchronous rotating frame is thus a nonholonomic frame.

Let the transformation tensor between the old and new coordinates be $C_{\alpha'}{}^{\alpha}$. Then

$$i^{\alpha} = C_{\alpha'}{}^{\alpha} i^{\alpha'} \tag{26}$$

where $i^{\alpha'}$ denotes the new currents or velocities. Since the power or energy in the system is *invariant* regardless of the particular set of variables, we get

$$P = e_\gamma i^\gamma = e_{\gamma'} i^{\gamma'}. \tag{27}$$

It follows that

$$e_{\gamma'} = C_{\gamma'}{}^{\gamma} e_\gamma \tag{28}$$

where $e_{\gamma'}$ denotes the new voltages or torques. When $C_{\alpha'}{}^{\alpha}$ is *non-singular*, it is possible to find its *inverse*. In general, however, $C_{\alpha'}{}^{\alpha}$ may be *singular*, and the invariancy of power calls for a relation given in Eq. 28 in which $C_{\gamma'}{}^{\gamma}$ is to be interpreted as the *transpose* of $C_{\alpha'}{}^{\alpha}$ given in Eq. 26.

Transforming Eq. 12 gives

$$\begin{aligned} e_{\gamma'} = C_{\gamma'}{}^{\gamma} e_\gamma = {} & (C_{\gamma'}{}^{\gamma} r_{\gamma\alpha} C_{\alpha'}{}^{\alpha}) i^{\alpha'} \\ & + (C_{\gamma'}{}^{\gamma} a_{\gamma\alpha} C_{\alpha'}{}^{\alpha}) \frac{di^{\alpha'}}{dt} + C_{\gamma'}{}^{\gamma} a_{\gamma\alpha} \left(\frac{\partial C_{\alpha'}{}^{\alpha}}{\partial x^{\beta'}} i^{\beta'} \right) i^{\alpha'} \\ & + C_{\gamma'}{}^{\gamma} [\alpha\beta, \gamma] (C_{\alpha'}{}^{\alpha} i^{\alpha'}) C_{\beta'}{}^{\beta} i^{\beta'}. \end{aligned} \tag{29}$$

Thus Lagrange's equation can be generalized to the *Boltzmann-Hamel* equation

$$\begin{aligned} e_{\gamma'} = {} & \frac{d}{dt} \left(\frac{\partial T'}{\partial i^{\gamma'}} \right) - \frac{\partial T'}{\partial x^{\gamma'}} \\ & + \frac{\partial F'}{\partial i^{\gamma'}} + C_{\gamma'}{}^{\gamma} C_{\alpha'}{}^{\alpha} \left(\frac{\partial C_\gamma{}^{\delta'}}{\partial x^\alpha} - \frac{\partial C_\alpha{}^{\delta'}}{\partial x^\gamma} \right) \frac{\partial T'}{\partial i^{\delta'}} i^{\alpha'} \end{aligned} \tag{30}$$

where

$$T' = \tfrac{1}{2} a_{\alpha'\beta'} i^{\alpha'} i^{\beta'} \tag{31}$$

$$F' = \tfrac{1}{2} r_{\alpha'\beta'} i^{\alpha'} i^{\beta'} \tag{32}$$

$$a_{\alpha'\beta'} = C_{\beta'}{}^{\beta} a_{\alpha\beta} C_{\alpha'}{}^{\alpha} \tag{33}$$

$$r_{\alpha'\beta'} = C_{\beta'}{}^{\beta} r_{\alpha\beta} C_{\alpha'}{}^{\alpha}. \tag{34}$$

In Eqs. 33 and 34, $C_{\alpha'}{}^{\alpha}$ is the same transformation tensor as that given in Eq. 26, and $C_{\beta'}{}^{\beta}$ corresponds to $C_{\gamma'}{}^{\gamma}$ in Eq. 28. In matrix notation, we have in general

$$Z' = C_t Z C \tag{35}$$

where Z denotes the old (original) impedance matrix, and Z' denotes the new impedance matrix. Substituting Eqs. 33 and 34 into Eq. 29 and introducing the *affine connection* give

$$e_{\gamma'} = r_{\gamma'\alpha'} i^{\alpha'} + a_{\gamma'\alpha'} \frac{di^{\alpha'}}{dt} + \Gamma_{\alpha'\beta',\gamma'} i^{\alpha'} i^{\beta'} \tag{36}$$

where the ... *connection* is obtained by identifying the sum of the third and fourth terms on the right side of Eq. 29 with the third term on the right side of Eq. 36, or

$$\Gamma_{\alpha'\beta',\gamma'} = a_{\gamma\alpha} \frac{\partial C_{\alpha'}{}^{\alpha}}{\partial x^{\beta'}} C_{\gamma'}{}^{\gamma} + [\alpha\beta, \gamma] C_{\alpha'}{}^{\alpha} C_{\beta'}{}^{\beta} C_{\gamma'}{}^{\gamma}. \tag{37}$$

Note also that

$$\Gamma_{\alpha'\beta',\gamma'} = [\alpha'\beta', \gamma'] + C_{\gamma'}{}^{\gamma} \left(\frac{\partial C_{\gamma}{}^{\delta'}}{\partial x^{\alpha}} - \frac{\partial C_{\alpha}{}^{\delta'}}{\partial x^{\gamma}} \right) a_{\delta'\beta'}. \tag{38}$$

5. THEORY OF HUNTING

Kron (7) first suggested the use of a *uniformly* rotating reference frame for the study of the hunting of one or two machines. An independent derivation of Kron's result was obtained by the author in the summer of 1951 while working as a consultant for the General Electric Company. The author submitted a comprehensive paper (6) including the hunting equation to the Institution of Electrical Engineers, London, in February 1952. At the urge of the author, Kron presented his "New Theory of Hunting" before the A.I.E.E. Summer General Meeting, June 1952. This paper later appeared in the *A.I.E.E. Transactions* (8). To quote from Kron:

> If, however, the hunting equations of a synchronous machine are established in such a manner that the armature reference frame merely rotates uniformly with respect to the stationary armature and does not oscillate, then the equations and equivalent circuit of the series-connected transmission lines become comparatively simple, unencumbered by additional generators in each mesh. . . . Any number of synchronous and induction machines and transmission lines then may be connected together as if they were merely stationary impedances.

It may be pointed out that in establishing an equivalent circuit it is necessary to use the forward and backward components or *Ku variables* (3, 5, 9) in the uniformly rotating reference frame instead of the direct and quadrature components, or *Park variables* (10).

In order to simplify the discussion, we start from Eq. 16 for the holonomic frame with $p = d/dt$. Thus Eq. 16 becomes

$$e_m = R_{mn} i^n + p\phi_m. \tag{39}$$

Transforming Eq. 39 to the nonholonomic frame (uniformly rotating frame) gives

$$e_{m'} = R_{m'n'} i^{n'} + p'\phi_{m'} \tag{40}$$

where p' denotes the *absolute* derivative and can be given by

$$p' = p + \gamma(p\theta_2). \tag{41}$$

This symbol γ denotes the "rotation" tensor which comprises the elements $+j$ and $-j$ in connection with the f (forward) and b (backward) components. When only one machine is hunting, $(p\theta_1)$ represents the actual angular velocity of the rotor, and $(p\theta_2) = \omega$ represents the fundamental angular velocity (frequency) of the generating system which is assumed uniform.

Under hunting conditions Eq. 40 gives

$$\Delta e_{m'} = R_{m'n'}\Delta i^{n'} + p'\Delta\phi_{m'} - b_{m'}p'(\Delta\theta) \tag{42}$$

where $b_{m'}$ is defined in Appendix I. The torque equation gives

$$\Delta\tau = \gamma i^{n'}(\Delta i^F) + b_{m'}\Delta i^{n'} + (Jp^2 + Bp - \gamma b_{m'}i^{n'})(\Delta\theta). \tag{43}$$

In Appendix I, $\phi_f = i^F + L_1 i^f + L_2 i^b$, and $\phi_b = i^B + L_1 i^b + L_2 i^f$, where $i^F = i^B = i^{Fd}/\sqrt{2}$, i^{Fd} denotes the direct-axis field excitation, and i^f and i^b are the f, b components.

Part III

6. THE QUASI-HOLONOMIC FRAME

In the quasi-holonomic reference frame we shall keep the rotor currents and voltages and the mechanical coordinates (θ and ω) as they are and transform the stator currents and voltages to a synchronously rotating reference frame. According to the revolving-field theory (**3, 4, 5, 6**), the transformed components are the forward and backward (f, b) components (or *Ku variables*). According to the two-reaction theory developed by Blondel, Doherty, Nicle, and Park (**10**), the transformed components are the direct and quadrature (d, q) components (or *Park variables*). On the stationary reference frame, the phase voltages and currents of a three-phase synchronous machine can be transformed to their symmetrical (positive, negative, and zero sequence) components or to the alpha and beta (α, β) components. (The *instantaneous* symmetrical (1, 2) components are termed *Lyon variables* and the corresponding (α, β) components are termed *Clarke variables*). For a two-phase machine, the armature phase currents and voltages can be labelled (α, β) currents and voltages. The relation between the symmetrical (1, 2) components and the (α, β) components is similar to the relation between the (f, b) components and the (d, q) components.

In the following sections we designate the indices $\alpha''\beta''$ as the new indices for the quasi-holonomic reference frame similar to the use of $\alpha'\beta'$ as the indices for the nonholonomic reference frame. Also we use the indices $m''n''$ to denote the electrical quantities and the indices $u''v''$

to denote the mechanical quantities. We introduce the indices $r''s''$ to denote the rotor quantities alone. However, just as the indices $\alpha''\beta''$ include both $m''n''$ and $u''v''$, it is understood that the indices $m''n''$ include the indices $r''s''$. In the quasi-holonomic reference frame here chosen, we let $e_{r''} = e_r$, $i^{s''} = i^s$, $R_{r''s''} = R_{rs}$, and $L_{r''s''} = L_{rs}$. Similarly, we let $e_{u''} = e_u$, $i^{v''} = i^v$, $J_{u''v''} = J_{uv}$, and $B_{u''v''} = B_{uv}$.

Let the transformation tensor be

$$C_{m''}{}^{m} = f(x^u), \qquad C_{r''}{}^{r} = \delta_{r''}{}^{r}, \qquad \text{and} \qquad C_{u''}{}^{u} = \delta_{u''}{}^{u}. \tag{44}$$

Then

$$i^m = C_{m''}{}^{m} i^{m''}, \qquad i^r = \delta_{r''}{}^{r} i^{r''} = i^{r''}, \qquad \text{and} \qquad i^u = \delta_{u''}{}^{u} i^{u''} = i^{u''}. \tag{45}$$

By using double primes instead of single primes in Eq. 30 we get the *equation of motion* as

$$e_{\gamma''} = \frac{d}{dt}\left(\frac{\partial T''}{\partial i^{\gamma''}}\right) - \frac{\partial T''}{\partial x^{\gamma''}} + \frac{\partial F''}{\partial i^{\gamma''}} + C_{\gamma''}{}^{\gamma} C_{\alpha''}{}^{\alpha}\left(\frac{\partial C_{\gamma}{}^{\delta''}}{\partial x^{\alpha}} - \frac{\partial C_{\alpha}{}^{\delta''}}{\partial x^{\gamma}}\right)\frac{\partial T''}{\partial i^{\delta''}} i^{\alpha''} \tag{46}$$

where T'' and F'' are defined similar to Eqs. 31 and 32 with double primes replacing single primes. From an equation similar to Eq. 29 we get the *equation of voltage* in a quasi-holonomic frame as

$$e_{m''} = R_{m''n''} i^{n''} + a_{m''n''}\frac{di^{n''}}{dt} + \frac{\partial a_{m''n''}}{\partial x^{u''}} i^{n''} i^{u''} + C_{m''}{}^{m}\frac{\partial C_{m}{}^{k''}}{\partial x^{u''}} a_{k''n''} i^{n''} i^{u''} \tag{47}$$

where

$$R_{m''n''} = C_{n''}{}^{n} R_{mn} C_{m''}{}^{m}, \qquad \text{and} \qquad a_{m''n''} = C_{n''}{}^{n} a_{mn} C_{m''}{}^{m}. \tag{48}$$

In Appendix II it is shown that for the quasi-holonomic frame chosen $R_{m''n''} = R_{mn}$, and $a_{m''n''} =$ constant, and hence $\partial a_{m''n''}/\partial x^{u''} = 0$.

The *equation of torque* in a quasi-holonomic frame is given by

$$e_{u''} = R_{u''v''} i^{v''} + a_{u''v''}\frac{di^{v''}}{dt} - \frac{1}{2}\frac{\partial a_{m''n''}}{\partial x^{u''}} i^{m''} i^{n''} - C_{n''}{}^{n}\frac{\partial C_{n}{}^{k''}}{\partial x^{u''}} a_{k''m''} i^{m''} i^{n''}. \tag{49}$$

Note again that for the quasi-holonomic frame chosen $R_{u''v''} = R_{uv}$, $a_{u''v''} = a_{uv}$, and $a_{m''n''} =$ constant so that $\partial a_{m''n''}/\partial x^{u''} = 0$.

In Appendix II we have shown that in terms of forward and backward (f, b) components Eq. 47 gives

$$e_f = R_{ff}i^f + \left(L_{ff}\frac{di^f}{dt} + L_{fb}\frac{di^b}{dt} + L_{fF}\frac{di^{Fd}}{dt}\right) + j(L_{ff}i^f + L_{fb}i^b + L_{fF}i^{Fd})\omega \quad (50)$$

$$e_b = R_{bb}i^b + \left(L_{bb}\frac{di^b}{dt} + L_{bf}\frac{di^f}{dt} + L_{bF}\frac{di^{Fd}}{dt}\right) - j(L_{bb}i^b + L_{df}i^f + L_{bF}i^{Fd})\omega \quad (51)$$

$$e_{Fd} = R_{FF}i^{Fd} + \left(L_{FF}\frac{di^{Fd}}{dt} + L_{Ff}\frac{di^f}{dt} + L_{Fb}\frac{di^b}{dt}\right) \quad (52)$$

where $\omega = i^{u''}$, and

$$R_{ff} = R_{bb} = R_1, \qquad L_{ff} = L_{bb} = L_1, \qquad L_{fb} = L_{bf} = L_2,$$
$$R_{FF} = R_3 = R_{Fd}, \qquad L_{FF} = L_3 = L_{Fd},$$
$$L_{fF} = L_{Ff} = L_{Fb} = L_{bF} = L_4.$$

Note that in case we have more than one winding on the rotor, Eq. 47 would contain the equation of voltage for the rotor as

$$e_{r''} = R_{r''n''}i^{n''} + a_{r''n''}\frac{di^{n''}}{dt} + \frac{\partial a_{r''n''}}{\partial x^{u''}}i^{n''}i^{u''}. \quad (53)$$

In the case of a synchronous machine with a direct-axis field winding only, the impedance matrix is

$$Z_{m''n''} = \begin{array}{c|ccc} & f & b & Fd \\ \hline f & R_1 + (p + j\omega)L_1 & (p + j\omega)L_2 & (p + j\omega)L_4 \\ b & (p - j\omega)L_2 & R_1 + (p - j\omega)L_2 & (p - j\omega)L_4 \\ Fd & pL_4 & pL_4 & R_3 + pL_3 \end{array} \quad (54)$$

The equation of torque (49) becomes, for $p = d/dt$,

$$\begin{aligned} \tau &= (B + Jp)\omega + i^f(j\phi_b) - i^b(j\phi_f) \\ &= (B + Jp)\omega + j(i^f - i^b)L_4i^{Fd} + jL_2(i^fi^f - i^bi^b) \\ &= (B + Jp)\omega - i^q(L_{ad}i^{Fd}) - (L_d - L_q)i^qi^d \\ &= (B + Jp)\omega - i^q\phi_d + i^d\phi_q \end{aligned} \quad (55)$$

where $L_{ad} = \sqrt{2}L_4$, $(L_d - L_q) = 2L_2$, $i^q = -j(i^f - i^b)/\sqrt{2}$, and $i^d = (i^f + i^b)/\sqrt{2}$. Note that $(L_d + L_q) = 2L_1$, where L_d and L_q are the direct and quadrature axis inductances. The flux-linkages are defined as follows:

$$\phi_f = L_1 i^f + L_2 i^b + L_4 i^{Fd} \tag{56}$$

$$\phi_b = L_1 i^b + L_2 i^f + L_4 i^{Fd} \tag{57}$$

$$\phi_d = L_d i^d + L_{ad} i^{Fd} \tag{58}$$

$$\phi_q = L_q i^q. \tag{59}$$

7. A FOUR-DIMENSIONAL PROBLEM

We shall consider Eqs. 50, 51, 52 and 55 as simultaneous equations of an electromechanical system with four coordinates x^f, x^b, x^{Fd}, and θ, and their velocities (or currents) i^f, i^b, i^{Fd}, and ω. We have been able to reduce the angle-varying parameters to constant parameters. However, if ω is considered as a variable, Eqs. 50 and 51 contain terms that are products of i^f, i^b, or i^{Fd}, and ω. Similarly Eq. 55 contains terms that are products of i^f, or i^b, and i^{Fd}, and of $(i^f)^2$ and $(i^b)^2$. We shall introduce the following notations:

$$v^1 = i^f, \quad v^2 = i^b, \quad v^3 = i^{Fd}, \quad \text{and} \quad v^4 = \omega \tag{60}$$

$$f_1 = e_f, \quad f_2 = e_b, \quad f_3 = e_{Fd}, \quad \text{and} \quad f_4 = \tau. \tag{61}$$

The constants R_1, R_3, L_1, L_2, L_3, and L_4 are as defined before. We shall also keep the constants B and J for the mechanical coordinate equation. Thus we have

$$f_1 = L_1 \frac{dv^1}{dt} + L_2 \frac{dv^2}{dt} + L_4 \frac{dv^3}{dt} + R_1 v^1 + j(L_1 v^1 + L_2 v^2 + L_4 v^3) v^4 \tag{62}$$

$$f_2 = L_2 \frac{dv^1}{dt} + L_1 \frac{dv^2}{dt} + L_4 \frac{dv^3}{dt} + R_1 v^2 - j(L_2 v^1 + L_1 v^2 + L_4 v^3) v^4 \tag{63}$$

$$f_3 = L_4 \frac{dv^1}{dt} + L_4 \frac{dv^2}{dt} + L_3 \frac{dv^3}{dt} + R_3 v^3 \tag{64}$$

$$f_4 = J \frac{dv^4}{dt} + B v^4 + j L_4 (v^1 - v^2) v^3 + j L_2 [(v^1)^2 - (v^2)^2]. \tag{65}$$

Since these differential equations are nonlinear, we cannot use the method of Laplace transforms or Heaviside's operational calculus. Let p denote d/dt only. Then the above equations become, for $w = pv$,

$$f_1 = L_1w^1 + L_2w^2 + L_4w^3 + R_1v^1 + j(L_1v^1 + L_2v^2 + L_4v^3)v^4 \quad (66)$$

$$f_2 = L_2w^1 + L_1w^2 + L_4w^3 + R_1v^2 - j(L_2v^1 + L_1v^2 + L_4v^3)v^4 \quad (67)$$

$$f_3 = L_4w^1 + L_4w^2 + L_3w^3 + R_3v^3 \quad (68)$$

$$f_4 = Jw^4 + Bv^4 + jL_4(v^1 - v^2)v^3 + jL_2(v^1v^1 - v^2v^2). \quad (69)$$

For a non-salient-pole synchronous machine, $L_2 = 0$, the above equations simplify to

$$f_1 = L_1w^1 + L_4w^3 + R_1v^1 + j(L_1v^1 + L_4v^3)v^4 \quad (70)$$

$$f_2 = L_1w^2 + L_4w^3 + R_1v^1 - j(L_1v^2 + L_4v^3)v^4 \quad (71)$$

$$f_3 = L_4w^1 + L_4w^2 + L_3w^3 + R_3v^3 \quad (72)$$

$$f_4 = Jw^4 + Bv^4 + jL_4(v^1 - v^2)v^3. \quad (73)$$

Equations 66, 67, 68, and 69 can be transformed into the d, q components to give the following:

$$f_d = L_dw^d + L_{Fd}w^{Fd} + R_1v^d - L_qv^qv^4 \quad (74)$$

$$f_q = L_qw^q + R_1v^q + (L_{ad}v^{Fd} + L_dv^d)v^4 \quad (75)$$

$$f_3 = L_{ad}w^d + L_3w^{Fd} + R_3v^{Fd} \quad (76)$$

$$f_4 = Jw^4 + Bv^4 - v^qL_{ad}v^{Fd} - (L_d - L_q)v^qv^d \quad (77)$$

where $f_d = e_d$, $f_q = e_q$, $v^d = i^d$, $pv^d = w^d$, $v^q = i^q$, $pv^q = w^q$, $v^{Fd} = i^{Fd} = v^3$, $pv^{Fd} = w^{Fd}$, and $L_{ad} = \sqrt{2}L_4$, $R_{Fd} = R_3$, and $L_{Fd} = L_3$. For a non-salient-pole machine, $L_d = L_q$, and the term for reluctance torque disappears in Eq. 77.

8. THE TORQUE EQUATION

We shall investigate the torque equation given by Eq. 55, or Eq. 77. Let i^{Fd} remain constant during the mechanical oscillation. Let also

$$i^d = A\cos\theta, \qquad i^q = -A\sin\theta \quad (78)$$

where A is a constant and θ (or δ) is the angular displacement between a point on the shaft at any instant and a synchronously rotating frame. This is known as the *power angle*. Equation 55 then gives

$$\tau = B\frac{d\theta}{dt} + J\frac{d^2\theta}{dt^2} + K\sin\theta + K_2\sin 2\theta \quad (79)$$

where $K = AL_{ad}i^{Fd}$, and $K_2 = L_2A^2 = A^2(L_d - L_q)/2$. For a non-salient-pole machine, $L_2 = 0$, and hence $K_2 = 0$. Equation 79 is *nonlinear* on account of the presence of $\sin\theta$ and $\sin 2\theta$ terms. For $K_2 = 0$, Eq. 79 is similar to the equation for a pendulum with damping constant B and spring (stiffness) constant K. A more elaborate form of Eq. 79 was given as (**11**, **12**, **13**, **14**, **15**)

$$\tau = J\frac{d^2\theta}{dt^2} + B(1 - b\cos 2\theta)\frac{d\theta}{dt} + K\sin\theta + K_2\sin 2\theta. \tag{80}$$

As Eq. 80 and a simplified form of Eq. 79, with $K_2 = 0$, were treated in (**9**), we shall discuss the solution of Eq. 79 here. The constants J and K can be normalized to unity, after proper changes in the other constants and the time scale and the torque unit. The new equation becomes

$$\frac{d^2\theta}{dt_1^2} + B'\frac{d\theta}{dt_1} + \sin\theta + K_2'\sin 2\theta = \tau_0 + \tau_1 = \tau. \tag{81}$$

We have retained the symbol τ for torque in spite of the change of unit. Note that τ_0 denotes the original torque load, and τ_1 denotes the additional torque suddenly applied. The phase-plane equation for the phase plane (θ, ω) is

$$\frac{d\omega}{d\theta} = \frac{\omega'}{\omega} = \frac{\tau - K_2'\sin 2\theta - \sin\theta - B'\omega}{\omega}. \tag{82}$$

To find the saddle point, set $\omega'(= d\omega/dt)$ and ω to zero. Thus

$$\tau = K_2'\sin 2\theta_s + \sin\theta_s \tag{83}$$

where θ_s denotes the angle at which the *saddle point* is located. The slope at the saddle point is given by

$$S = -a \pm (a^2 - 2K_2'\cos 2\theta_s - \cos\theta_s)^{1/2} \tag{84}$$

where $a = B'/2$. If Eq. 80 is used as the starting point, $a = B'(1 - b\cos 2\theta_s)/2$. If θ is expressed in degrees, the terms $(-2K_2'\cos 2\theta_s - \cos\theta_s)$ must be divided by 57.3. For illustration, let $B' = 0.01$, $K_2' = 0.35$, and $\tau = 1.0$. Then Eq. 83 becomes $1 = 0.35\sin 2\theta_s + \sin\theta_s$ and θ_s is obtained as 90°. The slope at the saddle point is

$$\begin{aligned} S &= -0.005 \pm [(0.005)^2 + 0.70/57.3]^{1/2} \\ &= -0.005 \pm 0.110 = 0.105 \quad \text{or} \quad -0.115. \end{aligned} \tag{85}$$

Following the negative slope, the $\omega - \theta$ trajectory is traced backward. This gives the *separatrix* which separates the stable region from the unstable region. The area inside this separatrix is *stable*. Beyond the separatrix the system is *unstable* and the machine pulls out. The construction of the trajectory can be made either by the *isocline* method of the *acceleration-plane* method (**11**, **12**). Suppose that the separatrix trajectory reaches $\omega = 0$ at $\theta = \theta_0$. Then the initial load is[3]

$$\tau_0 = 0.35 \sin 2\theta_0 + \sin \theta_0. \tag{86}$$

The limiting value of the additional torque load is

$$\tau_1 = \tau - \tau_0 = 1 - \tau_0. \tag{87}$$

9. THE VOLTAGE EQUATION

Now if we assume that ω is constant in Eqs. 50, 51, and 52, the impedance matrix given in Eq. 54 can be utilized to write the following equations:

$$e_f = [R_1 + (p + j\omega)L_1]i^f + (p + j\omega)L_2 i^b + (p + j\omega)L_4 i^{Fd} \tag{88}$$

$$e_b = (p - j\omega)L_2 i^f + [R_1 + (p - j\omega)L_1]i^b + (p - j\omega)L_4 i^{Fd} \tag{89}$$

$$e_{Fd} = pL_4 i^f + pL_4 i^b + (R_3 + pL_3)i^{Fd}. \tag{90}$$

Since the above three equations are *linear*, superposition holds. We may put $e_{Fd} = 0$ and study the effect of suddenly applying e_f and e_b on i^f, i^b, and the transient component of i^{Fd}. From Eq. 90, for $e_{Fd} = 0$, we get, after transposing terms,

$$(R_3 + pL_3)i^{Fd} = - pL_4 i^f - pL_4 i^b. \tag{91}$$

Premultiplying Eq. 91 by pL_4 gives

$$pL_4(R_3 + pL_3)i^{Fd} = - p^2 L_4^2 (i^f + i^b). \tag{92}$$

Since ω is considered constant, premultiplying Eqs. 88 and 89 by $(R_3 + pL_3)$ and substituting into the new equations Eqs. 91 and 92 give

$$\begin{aligned}(R_3+pL_3)e_f = {} & \{(R_3+pL_3)[R_1+(p+j\omega)L_1] - p(p+j\omega)L_4^2\}i^f \\ & + [(R_3+pL_3)(p+j\omega)L_2 - p(p+j\omega)L_4^2]i^b\end{aligned} \tag{93}$$

$$\begin{aligned}(R_3+pL_3)e_b = {} & \{(R_3+pL_3)[R_1+(p-j\omega)L_1] - p(p-j\omega)L_4^2\}i^b \\ & + [(R_3+pL_3)(p-j\omega)L_2 - p(p-j\omega)L_4^2]i^f.\end{aligned} \tag{94}$$

[3] If τ_0 is given as 0.434 per unit, Eq. 86 gives θ_0 as 15°. For a total torque load of say $\tau = 0.95$ corresponding to $\tau_1 = 0.516$, the phase-plane trajectory can be constructed according to Eq. 82. Then $\omega(t)$ and $\theta(t)$ can be obtained from the trajectory by the relation $(\Delta\theta/\Delta t) = \omega_{av}$, where ω_{av} denotes the average angular velocity during the small interval.

We have eliminated one column and one row from the matrix (54) involving the field quantities and obtained the operational (or transform) impedance matrix

$$Z = \begin{array}{c|cc} & f & b \\ \hline f & R_1 + (p + j\omega)L_1(p) & (p + j\omega)L_2(p) \\ b & (p - j\omega)L_2(p) & R_1 + (p - j\omega)L_1(p) \end{array} \tag{95}$$

where

$$L_1(p) = L_1 - \frac{pL_4^2}{R_3 + pL_3}, \qquad L_2(p) = L_2 - \frac{pL_4^2}{R_3 + pL_3}. \tag{96}$$

For getting the *transient* inductance (or reactance), p is set to ∞. Hence

$$L_1' = L_1\left(1 - \frac{L_4^2}{L_1L_3}\right), \qquad L_2' = L_2\left(1 - \frac{L_4^2}{L_2L_3}\right). \tag{97}$$

The sum of $L_1(p)$ and $L_2(p)$ gives $L_d(p)$, and the difference between $L_1(p)$ and $L_2(p)$ gives $L_q(p)$. Thus

$$L_d(p) = L_d - \frac{pL_{ad}^2}{R_3 + pL_3}, \qquad L_q(p) = L_q. \tag{98}$$

For $p = \infty$, we get

$$L_d' = L_1' + L_2' = L_d\left(1 - \frac{L_{ad}^2}{L_dL_{Fd}}\right), \qquad L_q' = L_q \tag{99}$$

where $L_{Fd} = L_3$ and $L_{ad} = \sqrt{2}L_4$. If there are damper windings, $L_1(p)$ and $L_2(p)$ will have more complicated expressions, and so will be $L_d(p)$ and $L_q(p)$. The *subtransient* inductances are again obtained by setting p to ∞. The symbols used are L_1'', L_2'', L_d'', and L_q'' (see Appendix I).

In terms of d, q components with field coordinate eliminated, we have

$$Z = \begin{array}{c|cc} & d & q \\ \hline d & R_1 + pL_d(p) & -\omega L_q(p) \\ q & \omega L_d(p) & R_1 + pL_q(p) \end{array}. \tag{100}$$

The transient analysis of synchronous machines can then be made with the help of either Eq. 95 or Eq. 100.

CONCLUSIONS

An electromechanical system can be expressed by Lagrange's equation of motion in the holonomic frame. As the angle θ is the "fourth dimension," inductances which vary with θ have to be considered *nonlinear*. By transforming the system from the holonomic frame to the nonholonomic frame, the new equation of motion takes a more general form involving the presence of the *affine connection* which reduces to the *Christoffel symbol* in the holonomic frame. Theory of hunting can be developed based on the nonholonomic frame. If we keep the rotor quantities and the mechanical coordinate as they are, we get the *quasi-holonomic frame*. Then certain nonholonomic objects become tensors. By referring the stator quantities to the rotor, we have two sets of components: the *f*, *b* components (or *Ku variables*) and the *d*, *q* components (or *Park variables*). The equations of voltage involve the products of i^f, i^b, or i^{Fd}, and the angular velocity ω. Since we consider ω as a dependent variable in the "fourth equation" (see Eqs. 66–69), these equations 66 and 67 are *nonlinear*. The equation of torque involves the product of i^q and i^{Fd} in the case of a non-salient-pole machine and of i^d and i^q additionally in the case of a salient-pole machine. Thus Eq. 69 is *nonlinear*. (Only Eq. 68 is linear in form).

The equation of motion of an electromechanical system is in general *nonlinear* either in the holonomic frame, or in the quasi-holonomic frame, or in the nonholonomic frame. Consider the set of four equations (66–69) again for the quasi-holonomic frame, we find that the torque equation (69) alone gives the well-known pendulum equation (79) for a non-salient-pole machine. The solution previously obtained from a differential analyzer can now be obtained by the phase-plane method used in nonlinear analysis. The effect of saliency can be taken into account by the form shown in Eq. 80. Stability study can be made by finding the saddle point and constructing the separatrix. If one is only interested in electrical transients under the assumption that ω is constant, then Eqs. 66–68 form a set of three *linear* differential equations with $v^4 = \omega$ considered as a constant. Applying the principle of superposition and eliminating $i^{Fd} = v^3$, we get an operational (or transform) impedance matrix given by either Eq. 95 or Eq. 100. Transient analysis can thus be made by transforming the equations of voltage from the holonomic frame to the quasi-holonomic frame with great advantage. It is suggested that Eqs. 66–69 or 74–77 should be solved simultaneously for the general case where separate solutions obtained by dividing the four dimensions into two sets (3 and 1) are no longer valid or justified.

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APPENDIX I

Starting from the stationary frame, the armature (stator) phase equations can be given as follows:

$$R_1 i^1 + p\phi_1 = e_1 \tag{AI-1}$$

$$R_1 i^2 + p\phi_2 = e_2 \tag{AI-2}$$

where i^1 and i^2 denote the positive and negative sequence armature current components, ϕ_1 and ϕ_2 denote the positive and negative sequence flux-linkages, and e_1 and e_2 denote the positive and negative sequence voltage (terminal) components. The flux-linkages are (**5**):

$$\phi_1 = L_{ad} i^F e^{j\theta} + L_1 i^1 + L_2 i^2 e^{j2\theta} \tag{AI-3}$$

$$\phi_2 = L_{ad} i^B e^{-j\theta} + L_1 i^2 + L_2 i^1 e^{-j2\theta} \tag{AI-4}$$

where $\theta = \omega t$, L_1 denotes the average value of L_d and L_q, the direct and quadrature inductances of a salient-pole machine, L_2 denotes one half of the difference $(L_d - L_q)$, and L_{ad} is the mutual inductance between the armature phase and the direct-axis field. The field current components are defined by

$$i^F = \frac{i^{Fd} + ji^{Fq}}{\sqrt{2}}, \qquad i^B = \frac{i^{Fd} - ji^{Fq}}{\sqrt{2}} \tag{AI-5}$$

where i^{Fd} and i^{Fq} denote the field excitation currents in the direct and quadrature axes. The field flux-linkage components are similarly

$$\phi_F = \frac{L_{ad} i^{Fd} + jL_{Fq} i^{Fq}}{\sqrt{2}}, \qquad \phi_B = \frac{L_{ad} i^{Fd} - jL_{Fq} i^{Fq}}{\sqrt{2}}. \tag{AI-6}$$

Since i^{Fq} is zero, Eqs. AI-5 and AI-6 simplify to

$$i^F = i^B = \frac{i^{Fd}}{\sqrt{2}}, \qquad \phi_F = \phi_B = L_{ad} i^F = L_{ad} i^B. \tag{AI-7}$$

In the literature, the value of L_{ad} is often assigned as unity in the *per-unit* system.

Transforming Eqs. AI-1 and AI-2 into a rotating reference frame gives

$$R_1 i^f + p\phi_f + j\omega\phi_f = e_f \tag{AI-8}$$

$$R_1 i^b + p\phi_b - j\omega\phi_b = e_b \tag{AI-9}$$

where

$$\phi_f = \phi_F + L_1 i^f + L_2 i^b \tag{AI-10}$$

$$\phi_b = \phi_B + L_2 i^f + L_1 i^b \tag{AI-11}$$

$$i^f = i^1 e^{-j\theta}, \qquad i^b = i^2 e^{j\theta} \tag{AI-12}$$

$$e_f = e_1 e^{-j\theta}, \qquad e_b = e_2 e^{j\theta} \tag{AI-13}$$

$$\phi_f = \phi_1 e^{-j\theta}, \qquad \phi_b = \phi_2 e^{j\theta}. \tag{AI-14}$$

Note that for $i_{Fq} = 0$, and $L_{ad} = 1$, ϕ_F and ϕ_B are defined by Eqs. (AI-7) and may be replaced by i^F and i^B.

Assuming all field axes eliminated, the conventional equations of hunting are given as[4]

$$\delta e_f - (p + j\omega)\Delta\phi_F = R_i \delta i^f + (p + j\omega)(L_1'' \delta i^f + L_2'' \delta i^b) + j\phi_f p(\Delta\theta) \tag{AI-15}$$

$$\delta e_b - (p - j\omega)\Delta\phi_B = R_1 \delta i^b + (p - j\omega)(L_2'' \delta i^f + L_1'' \delta i^b) - j\phi_b p(\Delta\theta). \tag{AI-16}$$

From the torque equation

$$\tau = (Jp^2 + Bp)\theta - i^b j\phi_f + i^f j\phi_b = (Jp^2 + Bp)\theta - i^b B_f - i^f B_b \tag{AI-17}$$

where J denotes moment of inertia, B denotes friction constant, $B_f = j\phi_f$ and $B_b = -j\phi_b$, we get the equation of hunting torque

$$\Delta\tau = (Jp^2 + Bp)(\Delta\theta) - ji^b(\Delta\phi_F) + ji^f(\Delta\phi_B) + b_f \delta i^b + b_b \delta i^f \tag{AI-18}$$

where

$$b_f = -B_f + jL_1'' i^f - jL_2'' i^b = -j(\phi_f - L_1'' i^f + L_2'' i^b) \tag{AI-19}$$

$$b_b = -B_b + jL_2'' i^f - jL_1'' i^b = j(\phi_b - L_1'' i^b + L_2'' i^f). \tag{AI-20}$$

Transposing terms in Eq. AI-18 gives

$$\Delta\tau + ji^b(\Delta\phi_F) - ji^f(\Delta\phi_B) = b_b \delta i^f + b_f \delta i^b + (Jp^2 + Bp)(\Delta\theta). \tag{AI-21}$$

Note that the second and third terms on the left side of Eq. AI-21 are equivalent to $i^q \Delta i_{Fd}$, where $i^q = (-ji^f + ji^b)/\sqrt{2}$, and $\Delta\phi_F = \Delta\phi_B = \Delta i^{Fd}/\sqrt{2}$ for $L_{ad} = 1$.

The absolute differentials are related to the conventional changes as follows:

$$\delta i^f = \Delta i^f - ji^f(\Delta\theta) \tag{AI-22}$$

$$\delta i^b = \Delta i^b + ji^b(\Delta\theta). \tag{AI-23}$$

Similarly we have

$$\delta e_f = \Delta e_f - je_f(\Delta\theta) \tag{AI-24}$$

$$\delta e_b = \Delta e_b + je_b(\Delta\theta). \tag{AI-25}$$

Substituting Eqs. AI-22–AI-25 into Eqs. AI-15 and AI-16 gives

$$\Delta e_f - (p + j\omega)\Delta i^{Fd}/\sqrt{2} = [R_1 + (p + j\omega)L_1'']\Delta i^f + (p + j\omega)L_2'' \Delta i^b - (p + j\omega)b_f(\Delta\theta) \tag{AI-26}$$

$$\Delta e_b - (p - j\omega)\Delta i^{Fd}/\sqrt{2} = [R_1 + (p - j\omega)L_1'']\Delta i^b + (p - j\omega)L_2'' \Delta i^f - (p - j\omega)b_b(\Delta\theta) \tag{AI-27}$$

where b_f and b_b are defined in Eqs. AI-19 and AI-20. Similarly, Eq. AI-21 gives

$$\Delta\tau + i^q \Delta i^{Fd} = b_b \Delta i^f + b_f \Delta i^b + (Jp^2 + Bp - jb_b i^f + jb_f i^b)(\Delta\theta). \tag{AI-28}$$

Note that in using Eqs. AI-24 and AI-25, e_f should be replaced by $(R_1 i^f + j\omega\phi_f) = (R_1 i^f + \omega B_f)$ and e_b should be replaced by $(R_1 i^b - j\omega\phi_b) = (R_1 i^b + \omega B_b)$. The incremental field excitation current Δi^{Fd} represents $\Delta e_{Fd}/R_{Fd}$, where e_{Fd} denotes the direct-axis field excitation voltage and R_{Fd} denotes the field winding resistance. The effect of the field winding and the damper windings is taken account of in the use of "transient" or "subtransient" inductance L_1'' and L_2'' corresponding to L_d'' and L_q''.

[4] δi and δe are the *absolute* differentials and $\Delta\theta$ denotes the conventional change of angle. The symbols L_1'' and L_2'' are introduced to denote the "subtransient" inductances corresponding to L_d'' and L_q'' used in two-reaction theory.

APPENDIX II

Equations of Voltage for Transient Analysis

From Eqs. AI-1–AI-4 we get

$$e_1 = R_1 i^1 + p(L_1 i^1 + L_2 e^{j2\theta} i^2 + L_4 e^{j\theta} i^{Fd}) \tag{AII-1}$$

$$e_2 = R_1 i^2 + p(L_1 i^2 + L_2 e^{-j2\theta} i^1 + L_4 e^{-j\theta} i^{Fd}) \tag{AII-2}$$

where $L_4 = L_{ad}/\sqrt{2}$, and $i^{Fd} = \sqrt{2} i^F = \sqrt{2} i^B$. If we assign the index 3 to the field excitation current i^{Fd} in addition to the indices 1 and 2 for the positive and negative sequence components, we have

$$R_{11} = R_{22} = R_1, \qquad L_{11} = L_{22} = L_1 \tag{AII-3}$$

$$L_{12} = L_2 e^{j2\theta}, \qquad L_{21} = L_2 e^{-j2\theta}, \qquad L_{13} = L_4 e^{j\theta}, \qquad L_{23} = L_4 e^{-j\theta}. \tag{AII-4}$$

Note that L_{mn} is a function of θ for $m \neq n$, and is *not* a function of the current i^n. The direct-axis field or rotor equation is given by

$$e_3 = e_{Fd} = R_3 i^3 + p(L_3 i^3 + L_4 e^{-j\theta} i^1 + L_4 e^{j\theta} i^2) \tag{AII-5}$$

where $i^3 = i^{Fd}$ and $e_3 = e_{Fd}$. Thus we have

$$R_{33} = R_3 = R_{Fd}, \qquad L_{33} = L_3 = L_{Fd} \tag{AII-6}$$

$$L_{31} = L_4 e^{-j\theta}, \qquad L_{32} = L_4 e^{j\theta}. \tag{AII-7}$$

Note that L_{13} and L_{31} are conjugates, and L_{23} and L_{32} are conjugates. Similarly, L_{12} and L_{21} are conjugates.

Let the symmetrical components be transformed to the f, b components by

$$C_f{}^1 = \begin{array}{c|cc} & f & b \\ \hline 1 & e^{j\theta} & \\ 2 & & e^{-j\theta} \end{array} \tag{AII-8}$$

The field current component will remain the same. We shall use the index 3 for the original frame and the index F for the new frame in combination with the f, b components.

$$C_F{}^3 = \begin{array}{c|c} & F \\ \hline 3 & 1 \end{array} \tag{AII-9}$$

Then we have

$$C_m{}^n = \begin{array}{c|ccc} & f & b & F \\ \hline 1 & e^{j\theta} & & \\ 2 & & e^{-j\theta} & \\ 3 & & & 1 \end{array} \tag{AII-10}$$

Let the positive and negative sequence flux-linkages be defined by

$$\phi_1 = L_1 i^1 + L_2 e^{j2\theta} i^2 + L_4 e^{j\theta} i^3 = \phi_f e^{j\theta} \tag{AII-11}$$

$$\phi_2 = L_1 i^2 + L_2 e^{-j2\theta} i^1 + L_4 e^{-j\theta} i^4 = \phi_b e^{-j\theta}. \tag{AII-12}$$

Then it is seen that by Eqs. AII-8 and AII-9 or by Eq. AII-10 we get

$$\phi_f = L_1 i^f + L_2 i^b + L_4 i^{Fd} \tag{AII-13}$$

$$\phi_b = L_1 i^b + L_2 i^f + L_4 i^{Fd}. \tag{AII-14}$$

Thus we have

$$L_{ff} = L_{bb} = L_1, \qquad L_{fb} + L_{bf} = L_2, \qquad \text{and} \qquad L_{fF} = L_{bF} = L_4. \tag{AII-15}$$

Transforming Eqs. AII-1 and AII-2 to the rotating reference frame (5) gives[5]

$$e_f = R_1 i^f + (p + j\omega)\phi_f = R_1 i^f + p\phi_f + \omega B_f \tag{AII-16}$$

$$e_b = R_1 i^b + (p - j\omega)\phi_b = R_1 i^b + p\phi_b + \omega B_b \tag{AII 17}$$

where

$$B_f = j\phi_f, \qquad B_b = -j\phi_b. \tag{AII-18}$$

Note that

$$R_{ff} = R_{bb} = R_1. \tag{AII-19}$$

The field equation (AII-5) becomes

$$e_{Fd} = R_3 i^{Fd} + p(L_3 i^{Fd} + L_4 i^f + L_4 i^b). \tag{AII-20}$$

Thus we have

$$R_{FF} = R_3, \qquad L_{FF} = L_3, \qquad L_{Ff} = L_{Fb} = L_4. \tag{AII-21}$$

So by choosing a quasi-holonomic frame we have succeeded in transforming the equations of voltage to a new set of equations with constant as well as reciprocal coefficients. Whether these equations are linear or nonlinear, however, depends upon whether the angular velocity ω is considered as another dependent variable (in the fourth dimension) or a mere constant.

Equations AII-16 and AII-17 can be rewritten as follows:

$$e_f = R_1 i^f + (p + j\omega)(L_1 i^f + L_2 i^b + L_4 i^{Fd}) \tag{AII-22}$$

$$e_b = R_1 i^b + (p - j\omega)(L_1 i^b + L_2 i^f + L_4 i^{Fd}). \tag{AII-23}$$

In terms of d, q components these become

$$e_d = R_1 i^d + p(L_d i^d + L_{ad} i^{Fd}) - \omega(L_q i^q) = R_1 i^d + p\phi_d + \omega B_d \tag{AII-24}$$

$$e_q = R_1 i^q + p(L_q i^q) + \omega(L_d i^d + L_{ad} i^{Fd}) = R_1 i^q + p\phi_q + \omega B_q \tag{AII-25}$$

where

$$\phi_d = L_d i^d + L_{ad} i^{Fd}, \qquad \phi_q = L_q i^q \tag{AII-26}$$

and

$$B_d = -\phi_q, \qquad B_q = \phi_d. \tag{AII-27}$$

The field equation can then be rewritten as

$$e_{Fd} = R_3 i^{Fd} + p(L_3 i^{Fd} + L_{ad} i^d). \tag{AII-28}$$

If the angular velocity ω is assumed constant, the effect of any disturbance in the terminal voltages calling for a transient analysis of the electrical quantities can be studied by the equations of voltage given in terms of the f, b or d, q components on a rotating reference frame. The insertion of external networks and transmission lines will modify these equations. However, if these external networks and transmission lines are also referred to the rotating reference frame, and represented by A and B, then Eqs. AII-22 and AII-23 can be modified by adding A and B to R_1, respectively.

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[5] It is easy to verify that

$$p(\phi_f e^{j\theta}) = e^{j\theta}(p\phi_f + j\phi_f\omega) = e^{j\theta}(p + j\omega)\phi_f$$

$$p(\phi_b e^{-j\theta}) = e^{-j\theta}(p\phi_b - j\phi_b\omega) = e^{-j\theta}(p - j\omega)\phi_b$$

where $\omega = p\theta$.

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PENETRATION OF A ROD INTO A SEMI-INFINITE TARGET

BY

WILLIAM A. ALLEN [1] AND JAMES W. ROGERS [1]

ABSTRACT

The penetration of metal rods into semi-infinite metal targets has been investigated experimentally at velocities up to 0.3 cm/μsec. The rods were composed of Au, Pb, Cu, Sn, Al, and Mg; the targets were aluminum. Results are compared with predictions from the hydrodynamic theory of jet penetration. Basic assumptions of the hydrodynamic theory were used to determine an effective yield strength of the target-rod combination. Experimental results indicate that the effective yield strength is relatively independent of the strength of the rod. The hydrodynamic theory of penetration was determined to be generally acceptable except where the density of the jet is much greater than that of the target. A gold jet reveals a new effect of *secondary penetration* which results in penetration greater than that predicted by theory.

INTRODUCTION

This paper deals with the penetration of a metal rod into a semi-infinite metal target. The problem has been considered previously in connection with the shaped-charge effect (**1**),[2] the impact of cylinders against plates (**2**), and the mechanism of indentation in a hardness test (**3**). Impact velocities produced by steel shaped charges ranged up to 0.8 cm/μsec, the cylinders (**2**) were fired against plates at velocities up to 0.02 cm/μsec, and the hardness tests were conducted at essentially static conditions. A hydrodynamic theory has been introduced to explain the penetrations at shaped-charge velocities; the theories of elasticity and plasticity have been used in the low velocity range. The experiment discussed in this paper was designed to check the validity of the hydrodynamic theory of penetration.

HYDRODYNAMIC THEORY OF PENETRATION

The conventional theory (**1**, **4**, **5**, **6**) of the penetration of a jet of length l, velocity v, and density ρ_1 into a semi-infinite target of density ρ_2 can be readily deduced. Figure 1*a* represents a rod of length l and density ρ_1 traveling with velocity v toward a semi-infinite target of density ρ_2. Figure 1*b* illustrates the situation just after impact. Figure 1*c* indicates the actual penetration process where the bottom of the crater travels at velocity u. Figure 1*d* represents the termination of *primary* penetration; the jet has been converted entirely into a cylindrical shell which, in general, is traveling with respect to the target. If the movement of the jet material is toward the bottom of the crater,

[1] U. S. Naval Ordnance Test Station, China Lake, Calif.

[2] The boldface numbers in parentheses refer to the references appended to this paper.

the possibility exists that subsequent *secondary* penetration will occur. Assume that penetration begins at the initial impact, that the penetration rate u is constant, and that penetration ceases when the rod is used up. If initial transients, secondary penetration, and cavitation are neglected, the duration of the penetration event is given by

$$t = \frac{l}{(v - u)}, \tag{1}$$

the final crater depth $P = ut$ is given by

$$P = \frac{ul}{(v - u)}, \tag{2}$$

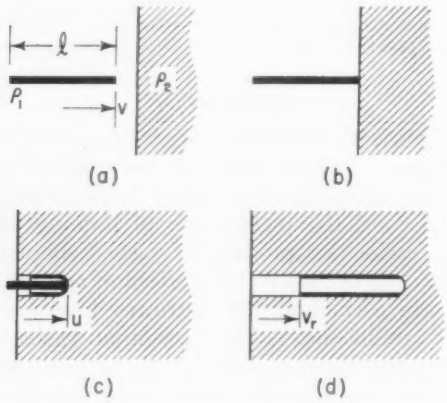

FIG. 1. Hydrodynamic penetration of a rod into a semi-infinite target.

and the stagnation stress is given by the modified Bernoulli's equation

$$\tfrac{1}{2}\rho_1(v - u)^2 = \tfrac{1}{2}\rho_2 u^2 + \phi, \tag{3}$$

where the term ϕ can be considered a dynamic yield strength of a solid target with respect to a fluid jet. The quantity ϕ is regarded as a strain-rate function involving the material strength of both rod and target. However, if ϕ, determined experimentally, should turn out to be relatively independent of jet material, then it can be inferred that the strength of the rod may be neglected and the resultant value of ϕ is a material characteristic of the target material alone. Equation 3 is valid for incompressible steady-state flow, but can be applied to cases of compressible isentropic steady-state flow. For the case of subsonic flow of a jet, Eq. 1 is a good approximation but is not exact

because the particle velocity component parallel to the trajectory is v only at infinity.

Solve Eq. 3 to obtain

$$u = \frac{\rho_1 v - [\rho_1^2 v^2 - 2(\rho_1 - \rho_2)(\frac{1}{2}\rho_1 v^2 - \phi)]^{1/2}}{\rho_1 - \rho_2}. \tag{4}$$

Substitute Eq. 4 into Eq. 2 to obtain

$$P = l\frac{-\rho_1 v + [\rho_1^2 v^2 - 2(\rho_1 - \rho_2)(\frac{1}{2}\rho_1 v^2 - \phi)]^{1/2}}{\rho_2 v - [\rho_1^2 v^2 - 2(\rho_1 - \rho_2)(\frac{1}{2}\rho_1 v^2 - \phi)]^{1/2}}. \tag{5}$$

In the limit of high velocity the effect of ϕ becomes negligible and Eqs. 4 and 5 approach the respective forms

$$u = \frac{v}{1 + (\rho_1/\rho_2)^{1/2}}, \tag{6}$$

$$P = l(\rho_1/\rho_2)^{1/2}. \tag{7}$$

The quantity ϕ in Eq. 3 can be defined for the critical case where $u = 0$ by means of the relation $\phi_0 = \frac{1}{2}\rho_1 v_0^2$. The critical velocity v_0 is the striking velocity that must be achieved before steady-state penetration can occur.

The phenomenon of jet penetration is related to the mechanism of indentation in a hardness test (**3**, **7**). A paper has appeared (**8**) in which an approximate solution was given for a rigid cylindrical punch penetrating an ideal plastic solid. Plastic yielding occurred when the mean pressure on the punch is about $2.8Y$ where Y is the yield stress of the material. A more recent analysis (**9**) for the case of a spherical punch penetrating an ideal plastic solid indicates that $\phi_0 = 2.6Y$. The *static* yield strength of 7075–T6 aluminum (**10**) is 0.00496 megabar. Data have been published (**11**) which indicate that the ultimate *dynamic* yield strength of duralumin of Brinell hardness 133 is 50 per cent higher than its static value. Thus, a theoretical value of ϕ_0 can be obtained. For future reference assume $\phi_0 = 0.019$ megabar for 7075–T6 aluminum.

The conventional hydrodynamic theory of a rod penetrating a target has never been adequately verified by shaped-charge jets because the assumed ideal conditions of the theory are difficult to achieve experimentally. A shaped-charge jet does not have a uniform cross section. In fact, the jet often consists of a stream of discrete fragments. Even when a jet remains continuous a velocity gradient exists along its length. Both the length l and velocity v of a shaped-charge jet are variables. Consequently, accurate experimental verification of Eq. 5 has not been reported previously.

EXPERIMENTAL RESULTS

A paper has appeared (12) that describes development of a hypervelocity precision impact instrument. This research tool has been used to check the theoretical penetration relation expressed by Eq. 5. A short metal wire segment approximately 0.020 in. in diameter was aligned on the trajectory of the instrument by means of a precision jig. This wire segment will subsequently be referred to as the *jet*. A 7075–T6 flat-nosed aluminum projectile, subsequently to be referred to as the *target*, was fired at the end of the jet (13). After impact the target travels down the range and is captured without appreciable

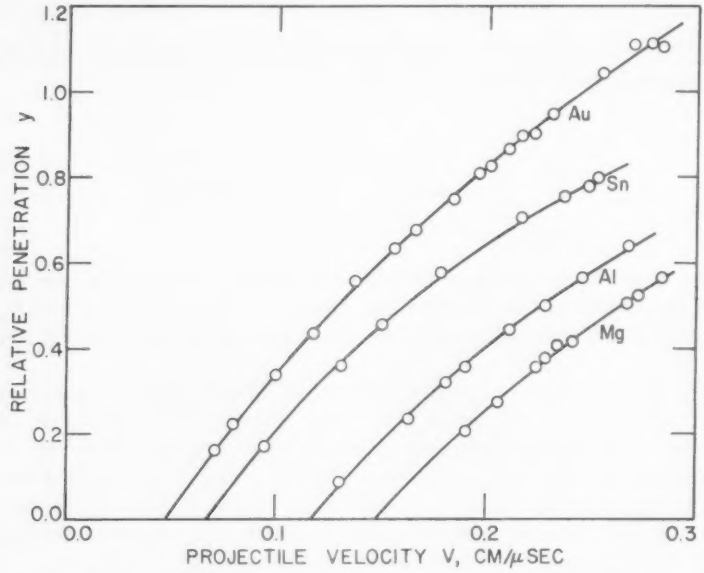

FIG. 2. Penetration of jets of different metals into 7075-T6 aluminum. The curves are empirical.

deformation in a cotton backstop. The target is then sectioned in order to determine the crater volume and the penetration. A cross section of a typical crater produced by a lead jet appears as Fig. 6 in (12).

Figure 2 displays experimental data for penetration of jets of various metals into 7075–T6 aluminum. The ordinate of Fig. 2 is relative penetration y defined by the relation

$$y = \frac{P}{l(\rho_1/\rho_2)^{1/2}}, \tag{8}$$

where P is the measured penetration, l is the length of the jet, and ρ_1, ρ_2 are the respective densities of jet and target. The solid curves

in Fig. 2 are empirical relations of the form

$$y = \left(\frac{\rho_2}{\rho_1}\right)^{1/2} \frac{a(v - v_0)}{(1 - a)(v - v_0) + v_0}, \tag{9}$$

where a and v_0 are adjustable parameters. A plot of Eq. 9 for the case of the lead jet appears as Fig. 7 in (12). The intercept v_0 on the x axis for each curve is a threshold velocity that must be achieved before steady-state penetration can occur. To each value of v_0 is associated the critical stagnation stress $\phi_0 = \frac{1}{2}\rho_1 v_0^2$.

Table I includes the physical properties of the jets used in the experi-

TABLE I.—*Physical and Experimental Properties of Jets.*

Metal	Length[a] l, cm	Diam d, cm	Density ρ_1, g/cc	Critical velocity v_0, cm/μsec	Adjustable parameter a
Au	0.467	0.0635	19.30	0.0479	0.90052
Pb	0.465	0.0737	11.09	0.0583	0.81329
Cu	0.432	0.0635	8.96	0.0672	0.81705
Sn	0.540	0.0711	7.30	0.0675	0.76894
Al	0.801	0.0787	2.70	0.1170	0.68072
Mg	0.796	0.0635	1.74	0.1481	0.64308

[a] Nominal value. The length of the jet varied slightly for different rounds.

ments. The last two columns of Table I present the adjustable parameters v_0 and a that appear in the empirical curves of Eq. 9. The experimental data of Fig. 2 can be displayed in an alternative and suggestive manner. Equation 2 written in the form

$$u = \frac{Pv}{(P + l)} \tag{10}$$

can be used to calculate the penetration velocity u as a function of the striking velocity v. Eliminate y between Eqs. 8 and 9 and substitute into Eq. 10 to obtain the relation

$$u = a(v - v_0). \tag{11}$$

Figure 3 is a plot of Eq. 11 for the four jet materials of Fig. 2. The values of the adjustable parameters a and v_0 in Table I were determined originally by fitting a straight line by the method of least squares to data plotted in the form of Fig. 3.

Substitute Eq. 11 into Eq. 3 and solve to obtain the relation

$$\phi = \tfrac{1}{2}\rho_1 v_0^2 - \rho_1 v_0 \left(1 - \frac{1}{a}\right) u + \tfrac{1}{2}\left[\rho_1 \left(1 - \frac{1}{a}\right)^2 - \rho_2\right] u^2. \tag{12}$$

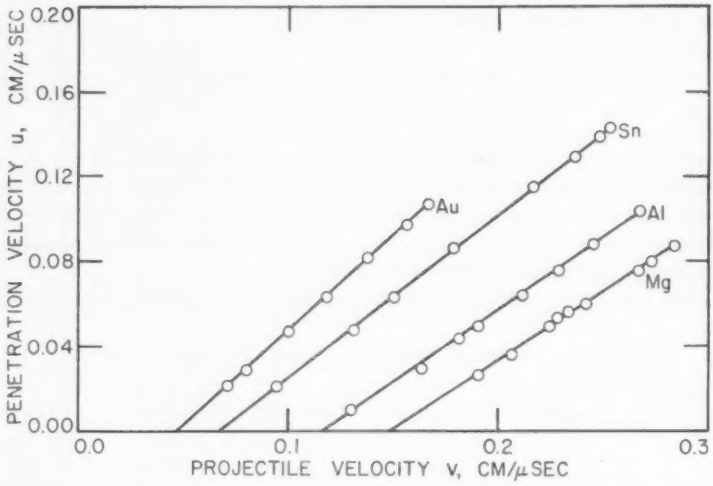

FIG. 3. Penetration velocity as function of projectile velocity for jets into 7075-T6 aluminum.

Too much physical significance should not be ascribed to Eq. 12 since its functional form is a simple consequence of the linear relation expressed by Eq. 11 and plotted in Fig. 3. Write Eq. 12 in the form

$$\phi(u) = \phi_0 + bu + cu^2, \tag{13}$$

where the constants ϕ_0, b, and c were calculated from Eq. 12 and are listed in Table II. Figure 4 is a plot of Eq. 13 where the values ϕ_0, b,

TABLE II.—*Parameters that Specity Dynamic Yield Strength ϕ in Eq. 13.*

Jet Material	ϕ_0	b	c
Gold	0.0221[a]	0.102[a]	−1.27[a]
Lead	0.0189	0.1487	−1.10
Copper	0.0202	0.1348	−1.17
Tin	0.0166	0.1481	−1.06
Aluminum	0.0185	0.1482	−1.09
Magnesium	0.0191	0.1430	−1.12
Average	0.0187±0.0012	0.144±0.005	−1.11±0.03

[a] Rejection of these data in averages was justified by Chauvenet's criterion.

c and the indicated dispersions were obtained from Table II. The average value $\phi_0 = 0.0187 \pm 0.0012$ megabar in Table II can be compared with the theoretical value $\phi_0 = 0.019$ megabar for 7075–T6 aluminum. The quantity $b = 0.144 \pm 0.005$ can be considered a measure of the strain-rate effect. The quantity $c = -1.11 \pm 0.03$ substituted into Eq. 13 forces the curve to reach a maximum value

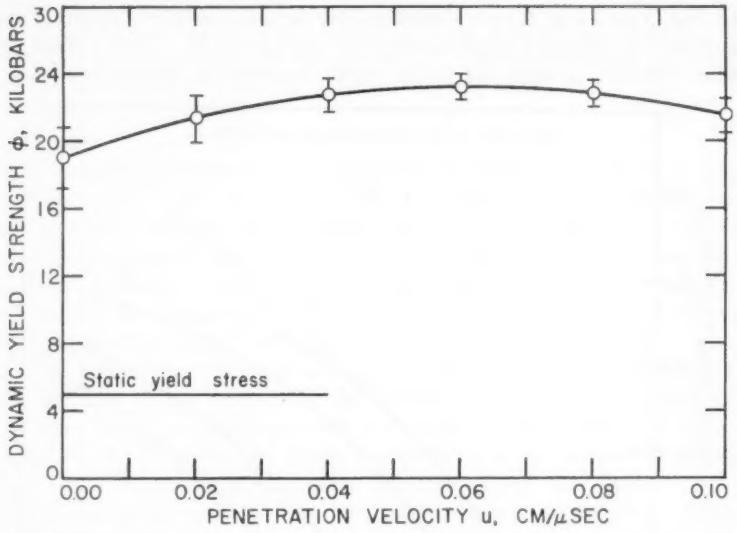

FIG. 4. Dynamic yield strength ϕ as function of penetration velocity for 7075-T6 aluminum target. Six jet materials were used.

$\phi = 1.2\phi_0$ at about $u = 0.06$ cm/μsec. The curve in Fig. 4 was discontinued at the penetration velocity $u = 0.10$ cm/μsec because data for aluminum and magnesium have not been obtained for velocities in excess of this limit and the data for gold beyond this value were suspect.

COMPARISON OF PENETRATION RESULTS WITH HYDRODYNAMIC THEORY

Agreement between the experimental data and hydrodynamic theory will now be investigated. In view of the relatively large dispersion in the curve of Fig. 4, a tentative average value of ϕ for all metals is assumed based on the relation

$$\langle\phi\rangle = \frac{\int_0^{0.1} \phi du}{\int^{0.1} du}. \tag{14}$$

Evaluation of Eq. 14 using the information of Table II leads to the value $\langle\phi\rangle = 0.0222$ megabar. Substitute this value in Eqs. 4 and 5 and evaluate to obtain the curves of Figs. 5 and 6. The average value $\phi = \langle\phi\rangle$ implies existence of an average threshold velocity for each metal specified by the relation

$$\langle\phi\rangle = \tfrac{1}{2}\rho_1\langle v_0\rangle^2. \tag{15}$$

Both Fig. 5 and Fig. 6 reveal systematic deviations from hydrodynamic theory in the case of a high density jet such as gold. The actual penetration velocity u appears to be more accurately represented by the

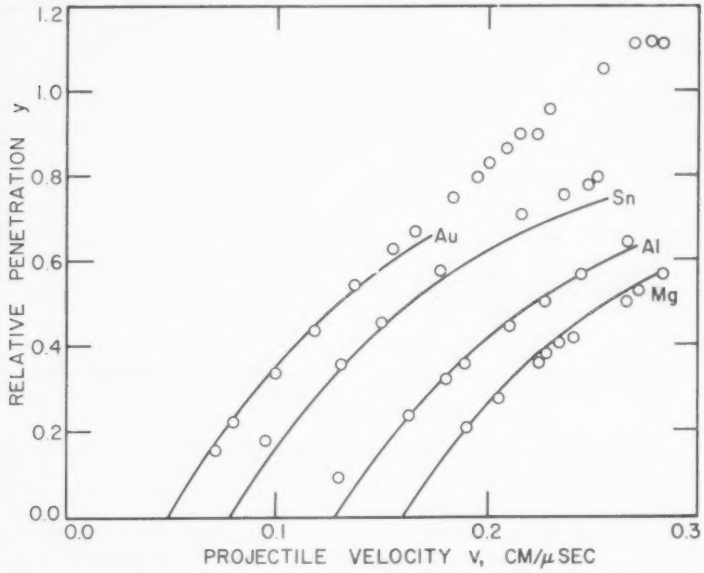

FIG. 5. Comparison of experimental penetration data with hydrodynamic theory. The curves are specified by Eq. 5 where $\phi = \langle\phi\rangle$ is the same constant for all metals.

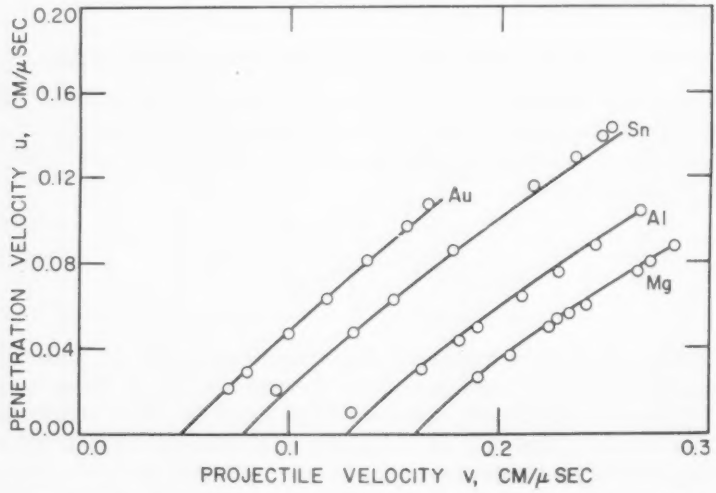

FIG. 6. Comparison of experimental penetration data with penetration theory. The curves are specified by Eq. 4 where $\phi = \langle\phi\rangle$ is the same constant for all metals.

straight lines of Fig. 3 than by the curves of Fig. 6. It should be recalled, however, that the four curves of Fig. 3 were specified by eight adjustable parameters while the curves of Fig. 6 were fixed by a single constant $\langle\phi\rangle$ which can be determined theoretically.

ANOMALOUS PENETRATION OF A GOLD JET

The anomalous penetration achieved in 7075–T6 aluminum with a high density jet such as gold can be explained by a new phenomenon of *secondary* or *residual penetration*. This effect is essentially different from a phenomenon previously reported (**14**) under the same name. This previous effect was explained by target flow subsequent to jet penetration; the discovered effect is a consequence of the difference in densities between the jet and the target.

With respect to a coordinate system traveling with the velocity u of the bottom of the crater the jet velocity is $v - u$. The speed of a free streamline is not affected by a change in direction, and thus the velocity of the jet reflected from the bottom of the crater is given by $u - v$. In the coordinate system of the target the velocity of the reflected jet is $2u - v$. When the jet has used up its entire length l its mass retains this velocity. The densities of jet and target are usually comparable in shaped-charge experiments and the residual velocity $2u - v$ is negligible compred with v. In the case of high density jets, however, the residual velocity is appreciable. The residual velocity $v_r = 2u - v$ can be calculated from Eq. 4, and is specified by the relation

$$v_r = \frac{2[\rho_1^2 v^2 - 2(\rho_2 - \rho_1)(\phi - \tfrac{1}{2}\rho_1 v^2)]^{1/2} - (\rho_2 + \rho_1)v}{\rho_2 - \rho_1}. \tag{16}$$

The threshold velocity v_0 is given by Eq. 15 where the brackets have been omitted without ambiguity. If $v_r > v_0$, secondary penetration can occur with magnitude given by Eq. 4 where v_r calculated from Eq. 16 is substituted for v. Equate Eq. 16 to v_0, eliminate $\phi = \langle\phi\rangle$ by means of Eq. 15, and solve to obtain

$$v_c = \frac{v_0(3\rho_1 + \rho_2)}{\rho_1 - \rho_2}. \tag{17}$$

Equation 17 specifies the critical velocity v_c at which secondary penetration begins to occur. This velocity occurs for the case of a gold jet into 7075–T6 aluminum at the value $v_c = 0.175$ cm/μsec; the curve in Fig. 7 illustrates the effect of secondary penetration in this case. The gold curves of Figs. 3 and 6 have not been extended past the critical velocity v_c. A gold jet is the only metal that causes secondary penetration in the velocity range studied.

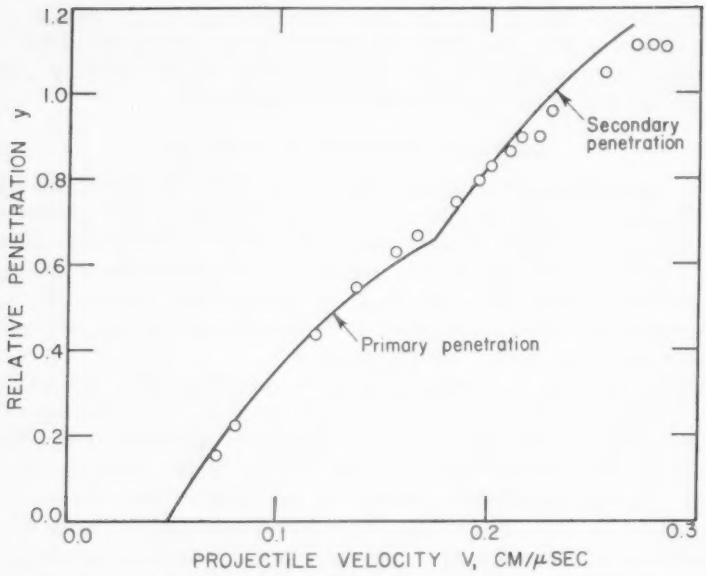

FIG. 7. Penetration of a gold jet into 7075-T6 aluminum. The line is a theoretical curve that illustrates the effect of secondary penetration.

Evidence exists that the anomalous penetration is not a result of cavitation. Penetration that results from cavitation would be independent of the length of the jet; its value is sometimes taken equal to the radius of the cylindrical crater (14). The new effect is proportional to jet length.

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TWO ANALOGIES IN SOLID CONTINUUM MECHANICS

BY

S. F. BORG [1]

SUMMARY [2]

This paper shows that the equations governing the behavior of moment and torque loaded thin plates which have small deflections are analogous to the equations of the two-dimensional linearized mathematical theory of isotropic elasticity. Because of the analogy, all solutions of two-dimensional plane-strain or plane-stress problems may be taken as solutions of a particular thin plate problem and vice versa.

A three-dimensional generalization of the above is also developed. In this form the theory draws an analogy between solutions in three-dimensional elasticity and those of thin plates subjected to lateral and shear loads as well as moments and torques.

The latter development is based on a tensoral generalization of the two-dimensional result to three dimensions. Because of this analogy all the known solutions for thin plates subjected to lateral loads represent possible solutions in a three-dimensional elastic structure,[3] and vice versa.

Examples of both analogies are given.

THE EQUATIONS OF THE TWO-DIMENSIONAL LINEARIZED MATHEMATICAL THEORY OF ISOTROPIC ELASTICITY

The fundamental tensors of two-dimensional linearized elasticity are the stress tensor, $\mathbf{T}$, given by[4]

$$\mathbf{T} = \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{pmatrix} \tag{1}$$

and the strain tensor, $\boldsymbol{\eta}$ given by

$$\boldsymbol{\eta} = \begin{pmatrix} e_x & \frac{1}{2}\gamma_{xy} \\ \frac{1}{2}\gamma_{yx} & e_y \end{pmatrix} = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & \dfrac{\partial v}{\partial y} \end{bmatrix}. \tag{2}$$

Note that these tensors are symmetrical. Also they are tensors in the sense that they satisfy a particular rotation of axes law, which

[1] Head, Civil Engineering Department, Stevens Institute of Technology, Hoboken, N. J.

[2] A preliminary form of the theory presented in this paper was given some years ago in the author's book "An Introduction to Matrix-Tensor Methods in Theoretical and Applied Mechanics," Ann Arbor, J. W. Edwards, 1956.

[3] An analogy between the plate solutions and membrane solutions is discussed at some length in S. TIMOSHENKO, "Theory of Plates and Shells," New York, McGraw-Hill Book Co., Inc., 1940.

[4] A list of the symbols used in this paper appears in the Appendix.

typically, is

$$\mathbf{T}' = \mathbf{RTR}^* \tag{3}$$

in which

T is the tensor referred to the x, y axes

T′ is the tensor referred to the x', y' axes (that is, the axes rotated about the origin)

R is the rotation matrix defined by

$$\mathbf{R} = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}. \tag{4}$$

R* is the adjoint matrix, that is, the matrix **R** with rows and columns interchanged, and

l_{ij} is the cosine of the angle between the i' and the j axes, where $i' = x', y'$ and $j = x, y$

The fundamental equations governing the behavior of a deformable body, in the two-dimensional linearized theory of elasticity, are:

stress equilibrium, body forces neglected

$$\operatorname{div} \mathbf{T} = 0 \tag{5}$$

or

$$\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{pmatrix} \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{pmatrix} = 0 \tag{6}$$

boundary conditions on stresses

$$T\bar{N} = \bar{\mathbf{x}} \tag{7}$$

or

$$\begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{pmatrix} \begin{pmatrix} l \\ m \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{y}} \end{pmatrix} \tag{8}$$

in which (l, m) are the direction cosines of a normal to the boundary and $\bar{\mathbf{x}}$, $\bar{\mathbf{y}}$, are the boundary stresses in the x, y directions.

Hooke's Law in its linearized isotropic form

$$\eta = \frac{1 + \nu}{E} T - \frac{\nu}{E} \mathcal{S}_1 \mathcal{S}_2 \tag{9}$$

and a

strain compatibility equation, in its linearized form

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = 2 \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}. \tag{10}$$

In Eq. 9, ν is Poisson's ratio, E is the isotropic modulus of elasticity, $\mathcal{I}_1$ is the trace of the stress tensor and $\mathcal{E}_2$ is the two-by-two unit matrix.

THE FUNDAMENTAL EQUATIONS OF SMALL DEFORMATION THIN PLATE THEORY

We may represent the loading, moments and torques acting on a typical element of a thin plate as in Fig. 1. The moments, M_{xx} and M_{yy}

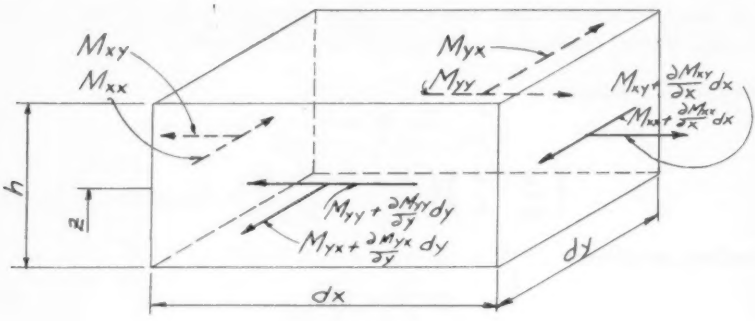

FIG. 1.

and the torques, $M_{xy} = -M_{yx}$ are given per unit length of side.

We may also define the curvatures, a typical value of which is

$$-\frac{1}{r_{xx}} = \frac{\partial^2 w}{\partial x^2} = \frac{\partial^{(\partial w/\partial x)}}{\partial x}. \tag{11}$$

Then (see footnote 2) we may show that the following tensors occur in the theory of moment and torque loaded thin plates with small deflections:

moment tensor

$$\mathfrak{M} = \begin{pmatrix} M_{xx} & -M_{xy} \\ M_{yx} & M_{yy} \end{pmatrix} \tag{12}$$

curvature tensor

$$\mathfrak{K} = \begin{bmatrix} -\frac{1}{r_{xx}} & \frac{1}{r_{xy}} \\ \frac{1}{r_{yx}} & -\frac{1}{r_{yy}} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} & \frac{1}{2}\left[\frac{\partial^{(\partial w/\partial y)}}{\partial x} + \frac{\partial^{(\partial w/\partial x)}}{\partial y}\right] \\ \frac{1}{2}\left[\frac{\partial^{(\partial w/\partial x)}}{\partial y} + \frac{\partial^{(\partial w/\partial y)}}{\partial x}\right] & \frac{\partial^2 w}{\partial y^2} \end{bmatrix}. \tag{13}$$

It will be noted that these are tensors, in the sense of Eq. 3, and are symmetrical. Also, because they are tensors they satisfy all the transformation, invariance and other relations governing tensors. In particular, because they are symmetrical they may be put in principal axis (that is, diagonal) form. Also, they satisfy the Mohr Circle transformation construction and the usual rotation relations for two-dimensional tensors.

It may be shown that the equations governing the action of thin, laterally loaded small deflected plates are (see footnote 2):

equilibrium equations

$$\operatorname{div} \mathfrak{M} = 0 \tag{14}$$

or

$$\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{pmatrix} \begin{pmatrix} M_{xx} & -M_{xy} \\ M_{yx} & M_{yy} \end{pmatrix} = 0 \tag{15}$$

boundary conditions

$$\mathfrak{M}\,\overline{N} = (\overline{\mathbf{M}}) \tag{16}$$

or

$$\begin{pmatrix} M_{xx} & -M_{xy} \\ M_{yx} & M_{yy} \end{pmatrix} \begin{pmatrix} l \\ m \end{pmatrix} = \begin{pmatrix} \overline{\mathbf{M}_{xx}} \\ \overline{\mathbf{M}_{yy}} \end{pmatrix} \tag{17}$$

in which $\overline{\mathbf{M}_{xx}}$ and $\overline{\mathbf{M}_{yy}}$ are the edge moments.

Relation Between Curvatures and Moments (Hooke's Law)

It may be shown (see footnote 2) that the following relation holds between $\mathfrak{K}$ and $\mathfrak{M}$

$$\mathfrak{K} = -\frac{1+\nu}{D(1-\nu^2)}\,\mathfrak{M} + \frac{\nu}{D(1-\nu^2)}\,\mathfrak{s}_1\mathcal{E}_2. \tag{18}$$

In Eq. 18 $\mathfrak{s}_1$ is the trace or first invariant of the moment tensor and D is the plate stiffness.

Finally, there is a

compatibility of curvature relation

$$\frac{\partial^2\left(-\frac{1}{r_{xx}}\right)}{\partial y^2} + \frac{\partial^2\left(-\frac{1}{r_{yy}}\right)}{\partial x^2} = 2\,\frac{\partial^2\left(\frac{1}{r_{xy}}\right)}{\partial x \partial y}. \tag{19}$$

THE ANALOGOUS QUANTITIES AND RELATIONS FOR THE TWO-DIMENSIONAL THEORY

In view of the foregoing, it follows that the following analogous quantities and relations may be set up for the two-dimensional isotropic elasticity-thin plate fields:

the tensors

$$\left.\begin{aligned} &T \sim \frac{\mathfrak{M}}{Kh^2} \\ \text{or} \qquad &\text{Eq. 1} \sim \frac{1}{Kh^2}\cdot\text{Eq. 12} \end{aligned}\right\} \tag{20}$$

K is a constant in the above. Also

$$\left.\begin{aligned} &\eta \sim h\mathfrak{R} \\ \text{or} \qquad &\text{Eq. 2} \sim h\cdot\text{Eq. 13} \end{aligned}\right\} \tag{21}$$

equilibrium equations

$$\left.\begin{aligned} &\operatorname{div}\mathbf{T} = 0 \sim \operatorname{div}\mathfrak{M} = 0 \\ \text{or} \qquad &\text{Eq. 5} \sim \text{Eq. 14} \end{aligned}\right\} \tag{22}$$

boundary conditions

$$\left.\begin{aligned} &\mathbf{T}\bar{N} = \bar{\mathbf{x}} \sim \mathfrak{M}\bar{N} = \overline{\mathbf{M}} \\ \text{or} \qquad &\text{Eq. 7} \sim \text{Eq. 16} \end{aligned}\right\} \tag{23}$$

Hooke's Law

$$\text{Eq. 9} \sim \text{Eq. 18} \tag{24}$$

compatibility condition

$$\text{Eq. 10} \sim \text{Eq. 19.} \tag{25}$$

We see therefore that there is a direct analogy between the two-dimensional equations of the linearized theory of isotropic two-dimensional elasticity and the governing equations for the thin, small deflection theory of moment and torque loaded plates.

Because of the analogy, it follows that solutions in the two-dimensional theory of isotropic elasticity may also be taken as analogous solutions in the theory of moment and torque loaded thin plates. In applying the analogy we compare a plate in the x, y plane (deflecting laterally in the z direction) to a two-dimensional deformable body loaded by stress or forces in the x, y directions. In particular, the analogous quantities are (using the subscript E for elasticity solutions and P for plate solutions):

Elasticity *Thin Plates*

$$\left.\begin{aligned}
\sigma_x &\sim \frac{M_{xx}}{Kh^2} \\
\sigma_y &\sim \frac{M_{yy}}{Kh^2} \\
\tau_{xy} = \tau_{yx} &\sim -\frac{M_{xy}}{K_1 h^2} = \frac{M_{yx}}{K_1 h^2} \\
e_x &\sim -\frac{h}{r_{xx}} \\
e_y &\sim -\frac{h}{r_{yy}} \\
\tfrac{1}{2}\gamma_{xy} = \tfrac{1}{2}\gamma_{yx} &\sim \frac{h}{r_{xy}} = \frac{h}{r_{yx}} \\
u_E &\sim -h\left(\frac{\partial w}{\partial x}\right)_P \\
v_E &\sim -h\left(\frac{\partial w}{\partial y}\right)_P
\end{aligned}\right\} \qquad (26)$$

Illustrative Example

As an illustration of the application of the analogy, we consider the following problem in the theory of elasticity, Fig. 2.[5]

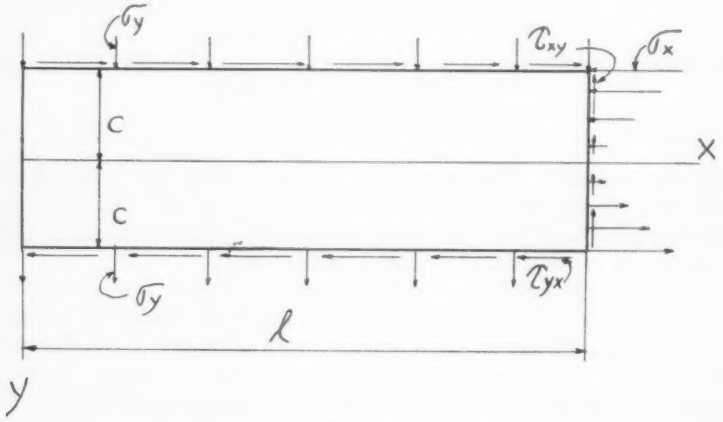

FIG. 2.

[5] S. TIMOSHENKO AND J. N. GOODIER, "Theory of Elasticity," second edition, New York, McGraw-Hill Book Co., Inc., 1951, p. 32.

The beam shown, with stresses

$$\left.\begin{aligned} \sigma_x &= d_5(x^2y - \tfrac{2}{3}y^3) \\ \sigma_y &= \tfrac{1}{3}d_5y^3 \\ \tau_{xy} &= -d_5xy^2, \end{aligned}\right\} \tag{27}$$

satisfies all the requirements of the two-dimensional linearized theory of elasticity. Along the top and bottom sides the normal stress σ_y is uniformly distributed. On the side $x = l$ the normal stresses follow a linear and cubic parabolic variation. The shear stress on the top and bottom is proportional to x and is parabolic on the face $x = l$.

Thus, the solution shown represents a thin plate solution, with

$$\left.\begin{aligned} M_{xx}/Kh^2 &\text{ in place of } \sigma_x \\ M_{yy}/Kh^2 &\text{ in place of } \sigma_y \end{aligned}\right\} \tag{28}$$

and

$$-\frac{M_{xy}}{Kh^2} \text{ in place of } \tau_{xy}. \tag{29}$$

From Hooke's Law the elasticity strains may be obtained as

$$\left.\begin{aligned} e_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) \\ e_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) \\ \gamma_{xy} &= \frac{2(1+\nu)}{E}\tau_{xy} \end{aligned}\right\} \tag{30}$$

and therefore the radii of curvature for the plate are known, using Eq. 26.

THE THREE-DIMENSIONAL FORM OF THE ANALOGY

We consider next a thin plate having small deflections and subjected to transverse and shear loads in addition to the edge moments and torques, see Fig. 3.

Then, it may be shown that the equilibrium equations for this structure are

$$\left.\begin{aligned} \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x &= 0 \\ -\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y &= 0 \\ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_x}{\partial y} + q &= 0 \end{aligned}\right\} \tag{31}$$

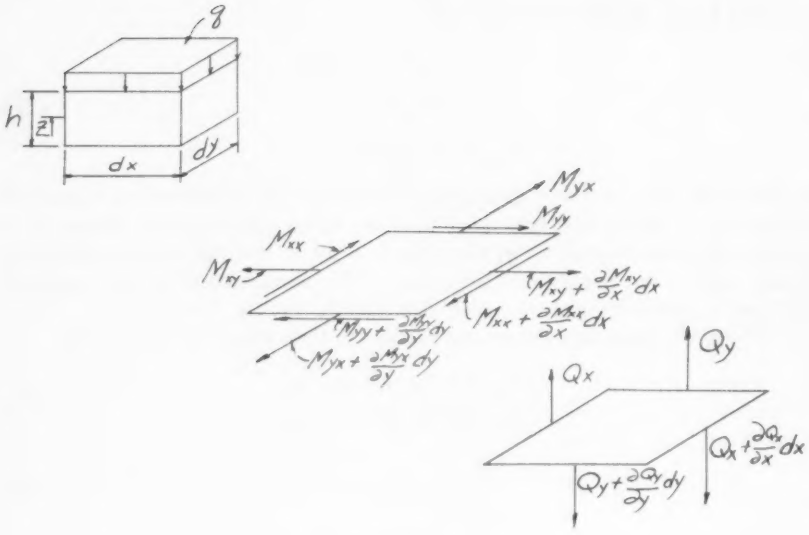

FIG. 3.

which is also given by

$$\operatorname{div} \mathfrak{M} = 0 \tag{32}$$

or

$$\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial z}\right)\begin{bmatrix} M_{xz} & -M_{xy} & -Q_x z \\ M_{yz} & M_{yy} & -Q_y z \\ -Q_x z & -Q_y z & -\frac{qz^2}{2} \end{bmatrix} = 0. \tag{33}$$

In the above a new tensor, $\mathfrak{M}$, defined by

$$\mathfrak{M} = \begin{bmatrix} M_{xz} & -M_{xy} & -Q_x z \\ M_{yz} & M_{yy} & -Q_y z \\ -Q_x z & -Q_y z & -\frac{qz^2}{2} \end{bmatrix} \tag{34}$$

has been introduced. This tensor represents the three-dimensional generalization of the two-dimensional form given in Eq. 12 and includes the effect of transverse and shear loads.

We are therefore led to introduce a generalized three-dimensional

form of $\mathfrak{K}$, as given in Eq. 13. This becomes, in its three-dimensional form,

$$\mathfrak{K} = \begin{pmatrix} -\frac{1}{r_{xx}} & \frac{1}{r_{xy}} & \frac{1}{r_{xz}} \\ \frac{1}{r_{yx}} & -\frac{1}{r_{yy}} & \frac{1}{r_{yz}} \\ \frac{1}{r_{zx}} & \frac{1}{r_{zy}} & -\frac{1}{r_{zz}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial^{(\partial w/\partial x)}}{\partial x} & \frac{1}{2}\left[\frac{\partial^{(\partial w/\partial y)}}{\partial x} + \frac{\partial^{(\partial w/\partial x)}}{\partial y}\right] & \frac{1}{2}\left[\frac{\partial^{(\partial w/\partial z)}}{\partial x} + \frac{\partial^{(\partial w/\partial x)}}{\partial z}\right] \\ \frac{1}{2}\left[\frac{\partial^{(\partial w/\partial x)}}{\partial y} + \frac{\partial^{(\partial w/\partial y)}}{\partial x}\right] & \frac{\partial^{(\partial w/\partial y)}}{\partial y} & \frac{1}{2}\left[\frac{\partial^{(\partial w/\partial z)}}{\partial y} + \frac{\partial^{(\partial w/\partial y)}}{\partial z}\right] \\ \frac{1}{2}\left[\frac{\partial^{(\partial w/\partial x)}}{\partial z} + \frac{\partial^{(\partial w/\partial z)}}{\partial x}\right] & \frac{1}{2}\left[\frac{\partial^{(\partial w/\partial y)}}{\partial z} + \frac{\partial^{(\partial w/\partial z)}}{\partial y}\right] & \frac{\partial^{(\partial w/\partial z)}}{\partial z} \end{pmatrix}. \quad (35)$$

It should be noted that $\mathfrak{M}$ and $\mathfrak{K}$ as formulated above are tensors in the sense of Eq. 3.

Because of the generalized nature of these tensors one is led to postulate a three-dimensional Hooke's Law as a direct generalization of the two-dimensional case, Eq. 18. This is given directly by

$$\mathfrak{K} = -\frac{1+\nu}{D(1-\nu^2)}\,\mathfrak{M} + \frac{\nu}{D(1-\nu^2)}\,\mathfrak{s}_1\mathcal{E}_3. \quad (36)$$

In this equation, $\mathfrak{K}$ and $\mathfrak{M}$ are given by Eqs. 35 and 34, $\mathfrak{s}_1$ is the trace of $\mathfrak{M}$ and $\mathcal{E}_3$ is the 3×3 unit matrix.

The boundary or edge conditions on the moments, torque and shears are

$$\mathfrak{M}\bar{N} = \begin{pmatrix} \overline{\mathbf{M}_{ii}} \\ \overline{\mathbf{Q}_i z} \end{pmatrix} \quad (37)$$

or

$$\begin{pmatrix} M_{xx} & -M_{xy} & -Q_x z \\ M_{yx} & M_{yy} & -Q_y z \\ -Q_x z & -Q_y z & -\frac{qz^2}{2} \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} \overline{\mathbf{M}_{xx}} \\ \overline{\mathbf{M}_{yy}} \\ \overline{\mathbf{Q}_i z} \end{pmatrix}. \quad (38)$$

In Eq. 38, l, m and n are the direction cosines of a normal to the edge of the plate at the central plane $z = 0$, hence $n = 0$. The right-hand side terms are the applied edge moment, torque and shear.

Also, the elements of the curvature tensor, $\mathfrak{K}$ satisfy the following six compatibility conditions

$$\left.\begin{aligned}
&\frac{\partial^2\left(-\frac{1}{r_{yy}}\right)}{\partial z^2}+\frac{\partial^2\left(-\frac{1}{r_{zz}}\right)}{\partial y^2}=2\,\frac{\partial^2\left(\frac{1}{r_{yz}}\right)}{\partial y\partial z}\\
&\frac{\partial^2\left(-\frac{1}{r_{zz}}\right)}{\partial x^2}+\frac{\partial^2\left(-\frac{1}{r_{xx}}\right)}{\partial z^2}=2\,\frac{\partial^2\left(\frac{1}{r_{zx}}\right)}{\partial z\partial x}\\
&\frac{\partial^2\left(-\frac{1}{r_{xx}}\right)}{\partial y^2}+\frac{\partial^2\left(-\frac{1}{r_{yy}}\right)}{\partial x^2}=2\,\frac{\partial^2\left(\frac{1}{r_{xy}}\right)}{\partial x\partial y}\\
&\frac{\partial^2\left(-\frac{1}{r_{zz}}\right)}{\partial x\partial y}=\frac{\partial}{\partial z}\left[\frac{\partial\left(\frac{1}{r_{yz}}\right)}{\partial x}+\frac{\partial\left(\frac{1}{r_{zx}}\right)}{\partial y}-\frac{\partial\left(\frac{1}{r_{xy}}\right)}{\partial z}\right]\\
&\frac{\partial^2\left(-\frac{1}{r_{xx}}\right)}{\partial y\partial z}=\frac{\partial}{\partial x}\left[\frac{\partial\left(\frac{1}{r_{zx}}\right)}{\partial y}+\frac{\partial\left(\frac{1}{r_{xy}}\right)}{\partial z}-\frac{\partial\left(\frac{1}{r_{yz}}\right)}{\partial x}\right]\\
&\frac{\partial^2\left(-\frac{1}{r_{yy}}\right)}{\partial z\partial x}=\frac{\partial}{\partial y}\left[\frac{\partial\left(\frac{1}{r_{xy}}\right)}{\partial z}+\frac{\partial\left(\frac{1}{r_{yz}}\right)}{\partial x}-\frac{\partial\left(\frac{1}{r_{zz}}\right)}{\partial y}\right]
\end{aligned}\right\}. \tag{39}$$

We now turn to the relations and equations of elasticity. In the three-dimensional theory of elasticity, the fundamental stress and strain tensors are,

$$\mathbf{T}=\begin{bmatrix}\sigma_x & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_y & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_z\end{bmatrix} \tag{40}$$

and

$$\eta=\begin{bmatrix}e_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz}\\ \frac{1}{2}\gamma_{yx} & e_y & \frac{1}{2}\gamma_{yz}\\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & e_z\end{bmatrix}$$

$$=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\\ \frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\\ \frac{1}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z}\end{bmatrix}. \tag{41}$$

The equilibrium equations in three-dimensional elasticity are

$$\operatorname{div} \mathbf{T} = 0 \tag{42}$$

or

$$\left(\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z}\right)\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = 0. \tag{43}$$

Also, Hooke's Law, the relation between the strain and stress tensor is

$$\boldsymbol{\eta} = \frac{1+\nu}{E}\mathbf{T} - \frac{\nu}{E}\,\mathcal{I}_1\mathcal{E}_3. \tag{44}$$

In Eq. 44, $\mathcal{I}_1$ is the trace of the stress tensor, $\mathcal{E}_3$ is the 3×3 unit matrix.

The boundary conditions on stresses is given by

$$\mathbf{T}\bar{N} = \bar{\mathbf{x}} \tag{45}$$

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}\begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{y}} \\ \bar{\mathbf{z}} \end{bmatrix} \tag{46}$$

and finally, the strains must satisfy the following six compatibility conditions,

$$\left.\begin{aligned} \frac{\partial^2 e_y}{\partial z^2} + \frac{\partial^2 e_z}{\partial y^2} &= 2\,\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \\ \frac{\partial^2 e_z}{\partial x^2} + \frac{\partial^2 e_x}{\partial z^2} &= 2\,\frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \\ \frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} &= 2\,\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\ \frac{\partial^2 e_x}{\partial x \partial y} &= \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) \\ \frac{\partial^2 e_z}{\partial y \partial z} &= \frac{\partial}{\partial x}\left(\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x}\right) \\ \frac{\partial^2 e_y}{\partial z \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y}\right) \end{aligned}\right\}. \tag{47}$$

By comparing the above, it becomes clear that an analogy may be drawn between the postulated thin plate relations, Eqs. 31–39 and the elasticity relations Eqs. 40–47.

This means that solutions in plate theory may represent solutions in the three-dimensional elasticity theory and vice-versa.

In Timoshenko's book (see footnote 3) many different plate solutions are given. Each of these corresponds to a particular solution in a three-dimensional stressed structure providing it satisfies all the relations, Eqs. 31–39. The analogous quantities are the following:

Elasticity		*Plate*
σ_x	$\sim$	$\dfrac{M_{xx}}{Kh^2}$
σ_y	$\sim$	$\dfrac{M_{yy}}{Kh^2}$
$\tau_{xy} = \tau_{yx}$	$\sim -$	$\dfrac{M_{xy}}{K_1h^2} = \dfrac{M_{yx}}{K_1h^2}$
σ_z	$\sim -$	$\dfrac{qz^2}{K_2h^2}$
$\tau_{xz} = \tau_{zx}$	$\sim -$	$\dfrac{Q_xz}{K_3h^2}$
$\tau_{yz} = \tau_{zy}$	$\sim -$	$\dfrac{Q_yz}{K_3h^2}$
u_E	$\sim -$	$h\left(\dfrac{\partial w}{\partial x}\right)_P$
v_E	$\sim -$	$h\left(\dfrac{\partial w}{\partial y}\right)_P$
w_E	$\sim -$	$h\left(\dfrac{\partial w}{\partial z}\right)_P$

(48)

The procedure may be summarized as follows:

1. Knowing $w(x, y)_P$ determine M_{xx}, M_{yy}, M_{xy}, Q_x and Q_y using the known plate relations. These are all functions of (x, y) only.
2. Using the values from (1), from Hooke's Law (Eq. 36) determine the $1/r$ terms. Note, these are now functions of (x, y, z).
3. Check the compatibility equations, Eq. 39. If these hold, then the analogous relations, Eq. 48, follow.

As an example, it is well known that the exact solution for a uniformly loaded elliptical plate with built-in edges is given by

$$w(x, y) = c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 \tag{49}$$

in which c is a constant and a, b are the major and minor elliptic radii, respectively. Using the known plate relations we obtain from Eq. 36 the various elements of $\mathfrak{R}$ which when substituted in Eq. 39, give 6 equations. The third, fourth, fifth and sixth of these are satisfied identically. The first and second will be satisfied if

$$\left.\begin{aligned} &a = b \text{ (that is, for a circle)} \\ \text{and if} \quad &\nu = 1. \end{aligned}\right\} \tag{50}$$

Using these we find the analogous stresses are given by

$$\sigma_x \sim -\frac{12M_{xx}}{h^2}$$

$$\sigma_z \sim \frac{6qz^2}{h^2}$$

$$\tau_{xz} \sim \frac{12Q_x z}{h^2} \text{ etc.}$$

Then using Eq. 44 we can obtain the strains and from these the deformations u, v and w; or, these could be obtained using Eq. 48.

CONCLUSION

Two separate analogies between elasticity theory-thin plate theory were developed.

In the first, two-dimensional elasticity solutions are shown to be analogous to thin plates with small deflections and torque and moment loaded only.

In the second, three-dimensional elasticity solutions are shown to be analogous to thin plates with small deflections and laterally loaded as well as torque and moment loaded.

Examples of both were given.

APPENDIX

Symbols Used in This Paper

$\mathbf{T}$ = stress tensor
$\boldsymbol{\eta}$ = strain tensor
σ_x = normal stress in x direction
σ_y = normal stress in y direction
σ_z = normal stress in z direction
$\tau_{yx} = \tau_{xy}$ = shear stress in x–y direction
$\tau_{zy} = \tau_{yz}$ = shear stress in y–z direction

$\tau_{zx} = \tau_{xz}$ = shear stress in z–x direction
e_x = normal strain in x direction
e_y = normal strain in y direction
e_z = normal strain in z direction
$\gamma_{xy} = \gamma_{yx}$ = shear strain in x–y direction
$\gamma_{yz} = \gamma_{zy}$ = shear strain in y–z direction
$\gamma_{zx} = \gamma_{xz}$ = shear strain in z–x direction
u = deformation in x direction
v = deformation in y direction
w = deformation in z direction, deflection in the z direction of thin plate
x, y, z = coordinate axes
h = plate thickness
ν = Poisson's ratio
E = Modulus of elasticity in tension-compression
ϑ_1 = the trace, the first invariant of the tensor
$\mathfrak{E}_3, \mathfrak{E}_2$ = unit matrix, $\begin{pmatrix} 100 \\ 010 \\ 001 \end{pmatrix}, \begin{pmatrix} 10 \\ 01 \end{pmatrix}$
M_{xx} = plate moment per unit length on a plane perpendicular to the x axis
M_{yy} = plate moment per unit length on a plane perpendicular to the y axis
$M_{xy} = -M_{yx}$ = plate torque in the x–y direction
q = lateral load per unit area on the plate
Q_x = plate shear force, per unit length, on a plane perpendicular to x axis
Q_y = plate shear force, per unit length, on a plane perpendicular to y axis
$\mathfrak{M}$ = moment tensor, defined by Eqs. 12, 34
$\mathfrak{K}$ = curvature tensor, defined by Eqs. 13, 35
$\frac{1}{r_{xx}}$ = typical curvature element of the curvature tensor
D = plate stiffness, $\frac{Eh^3}{12(1-\nu^2)}$
$\bar{\mathbf{x}}$ = boundary stress in x direction
$\bar{\mathbf{y}}$ = boundary stress in y direction
$\bar{\mathbf{z}}$ = boundary stress in z direction
$\overline{\mathbf{M}}_{ii}$ = boundary moment in the i direction
$\overline{\mathbf{M}}_{ij}$ = boundary torque in the i–j direction
$\bar{\mathbf{Q}}_i$ = boundary shear
K = a scale factor constant in the analogy relations between $\begin{Bmatrix} \mathbf{T} & \text{and} & \mathfrak{M} \\ \boldsymbol{\eta} & \text{and} & \mathfrak{K} \end{Bmatrix}$

ANALOGY OF NONLINEAR SYSTEMS TO CLASSICAL DYNAMICS *

BY

RUEY-WEN LIU [1] AND GILBERT H. FETT [2]

ABSTRACT

The two-dimensional autonomous system in the general form is considered. The solution is in terms of the phase trajectories in the phase plane. In this paper, a force field over the phase plane is formulated such that the path of a particle in this force field coincides with the path of the trajectories, provided the initial condition is properly chosen. Hence, the trajectories can be obtained by solving the path of the particle subject to the formulated force field. It is further developed that, for any given system, it is theoretically possible to formulate a *conservative* force field to give the same property. Consequently, it is found out that the trajectories can be solved either analytically by a simple substitution or graphically by means of the family of trajectories which are orthogonal to the trajectories of the original system. Various examples including the generalized Van der Pol equation are given gs illustrations.

I. INTRODUCTION

Equations under consideration are a pair of first order autonomous differential equations, defined as System (1), in the general form

$$\begin{aligned} \dot{x} &= p(x, y) \\ \dot{y} &= q(x, y), \end{aligned} \tag{1}$$

where the dot denotes the time derivative. It is assumed that the partial derivatives of $p(x, y)$ and $q(x, y)$ with respect to x and y exist and are continuous in the domain considered. If $p(x, y) = y$, then System (1) becomes a single differential equation of second order: $\ddot{x} = q(x, \dot{x})$.

If the independent variable time t is eliminated, System (1) can be written as

$$\frac{dy}{dx} = \frac{q(x, y)}{p(x, y)}. \tag{2}$$

Since the partial derivatives of $p(x, y)$ and $q(x, y)$ exist and are continuous, the Lefschetz condition is satisfied (1).[3] Consequently, the

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[1] University of Notre Dame, Notre Dame, Ind.

[2] Department of Electrical Engineering, University of Illinois, Urbana, Ill.

[3] The boldface numbers in parentheses refer to the references appended to this paper.

solution of (2) exists with the form

$$f(x, y; c) = 0, \tag{3}$$

where c is an arbitrary constant. Equation 3 can be represented as a family of curves with parameter c in the x–y plane which is called the *phase plane*. The curves in the phase plane corresponding to (3) are called the *phase trajectories*, or simply the *trajectories*. Equation 2 is the differential equation of these phase trajectories.

The purpose of this paper is to formulate a force field such that an analogy between

(*A*) the trajectories of System (1) in the phase plane, and

(*B*) the orbits of a particle with unit mass subject to the formulated force field are made. Hence the trajectories of System (1) can be obtained by solving the latter.

The method of analogy is given in Section II. After the analogy is made, several special classes of $p(x, y)$ and $q(x, y)$ are considered and solved in Sections III and IV, and the functions $p(x, y)$ and $q(x, y)$ in general are considered in Section V. Examples, such as the generalized Van der Pol equation, $\ddot{x} + f(x)\dot{x} + g(x) = 0$, are used as illustrations.

II. THE ANALOGY

When the time derivative of both sides of System (1) is taken, one obtains

$$\ddot{x} = \frac{\partial p}{\partial x} p + \frac{\partial p}{\partial y} q = P(x, y) \tag{4a}$$

and

$$\ddot{y} = \frac{\partial q}{\partial x} p + \frac{\partial q}{\partial y} q = Q(x, y). \tag{4b}$$

The right-hand sides of (4*a*) and (4*b*) are functions of x and y only, hence they can be defined as $P(x, y)$ and $Q(x, y)$.

Let the force field $\mathbf{F}(x, y)$ be defined as

$$\mathbf{F}(x, y) = \mathbf{i}P(x, y) + \mathbf{j}Q(x, y) = \mathbf{i}\ddot{x} + \mathbf{j}\ddot{y}. \tag{5}$$

By doing so, Eq. 5 becomes the equation of motion of a particle with unit mass subject to a force field whose x- and y-components are $P(x, y)$ and $Q(x, y)$. The force field so defined is called the *associated force field* of System (1), and Eq. 5 is called the *associated equation of motion*. To be distinguished from the trajectories, the paths of the unit mass particle are called the *orbits*. The *associated momentum field* $\mathbf{M}(x, y)$ of the unit mass particle is thus defined as

$$\mathbf{M}(x, y) = \mathbf{i}p(x, y) + \mathbf{j}q(x, y) = \mathbf{i}\dot{x} + \mathbf{j}\dot{y} \tag{6}$$

which is equivalent to System (1).

Since the phase trajectories of System (1) should satisfy (6), and since (6) is a special integral of (5), the solution of System (1) is the subset of the general solution of (5). Furthermore, this subset should also satisfy (6) at all times, in particular, at time zero. Therefore, the following analogy can be stated:

> (*A*) The family of the phase trajectories of System (1) is a subset of (*B*) the totality of the orbits of a unit mass particle subject to the associated force field of System (1). Furthermore, the initial condition of this subset should satisfy the associated momentum field.

It is noticed that, after the initial time, these trajectories also satisfy the associated momentum field.

By this analogy, a purely mathematical problem has been changed to a dynamical one. Since the physical quantities have been attributed to facets of the mathematical problem, intuition may enlighten the solution to the problem. Furthermore, for certain types of force field obtained by (5), the solution of System (1) may be obtained in a straightforward manner.

III. A CLASS OF p AND q

Consider the class of p and q of System (1) of which the associated equation of motion is integrable to the form

$$f(\dot{x}, \dot{y}; x, y; c) = 0, \tag{7}$$

where c is an arbitrary constant. Since (7) should be satisfied by all the orbits which are the solution of the associated equation of motion, it should also be satisfied by the trajectories. Hence the trajectories should satisfy (1) and (7) simultaneously. If (1) and (7) are independent, then $\dot{x}$ and $\dot{y}$ can be eliminated between them, and the phase trajectories of System (1) can be simply written out as

$$f[p(x, y), q(x, y); x, y; c] = g(x, y; c) = 0.$$

Hence the following theorem is obtained:

Theorem I: If the associated equation of motion of System (1) can be integrated to the form of (7), and if (7) and System (1) are independent, then the phase trajectories of System (1) can be obtained by a simple substitution.

The following examples will illustrate this point.

Central Force Fields (2)

If the components of the force field $P(x, y)$ and $Q(x, y)$ satisfy the condition

$$yP(x, y) = xQ(x, y), \tag{8}$$

then the θ-component of the force field vanishes identically. This force field is called the central force field. Consequently, the equation of motion can be integrated to give the equation of conservation of angular momentum, that is, in the rectangular coordinates,

$$x\dot{y} - \dot{x}y = c. \tag{9}$$

The associated force field of the following example is central.

Example 1. Consider the system of undamped harmonic oscillator:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -Ax, \end{aligned} \tag{10}$$

where A is any positive constant. The components of the associated force field are obtained from (4):

$$\begin{aligned} P(x, y) &= -Ax \\ Q(x, y) &= -Ay \end{aligned}$$

which satisfy (8). Hence the associated force field is a central one. Consequently, the angular momentum of the unit mass particle is conserved. The trajectories of System (10) thus are obtained by substituting (10) into (9):

$$Ax^2 + y^2 = -c = c'.$$

For positive A, the trajectories are ellipses in the phase plane, which are closed trajectories; hence System (10) will oscillate in such a way that the particle moves on the constant angular momentum curve.

Conservative Force Fields (3) *of Type I*

Suppose the components of the force field P and Q satisfy the relation

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{11}$$

which is equivalent to Curl $\mathbf{F}(x, y) = 0$, then the force field is conservative. Consequently, the equation of motion can be integrated to give the integral of energy:

$$\tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + V(x, y) = E, \tag{12}$$

where E is the total energy. The potential energy $V(x, y)$ is given by

$$V(x, y) = -\int_{x_0}^{x} P(x, y_0)dx - \int_{y_0}^{y} Q(x, y)dy \tag{13}$$

where (x_0, y_0) is the arbitrarily chosen reference point. Furthermore, if Eqs. 12 and 1 are independent which is defined as *Type I*, then the trajectories of System (1) can be obtained by a simple substitution. The following example will be given as an illustration.

Example 2: Consider the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x + Be^{kx} + A - (\tfrac{1}{2})ky^2 \end{aligned} \tag{14}$$

where A, B and k are any constants. From (4), the following components of the associated force field are obtained,

$$\begin{aligned} P(x, y) &= x + Be^{kx} + A - (\tfrac{1}{2})ky^2 \\ Q(x, y) &= y - Aky + (\tfrac{1}{2})k^2y^3 - kxy, \end{aligned}$$

which satisfy (11). Hence the associated force field is conservative, and the trajectories should satisfy (12) as well as (14). If the origin is chosen as the reference point, the potential field is obtained from (13):

$$V(x, y) = -(\tfrac{1}{2})x^2 - Ax + (\tfrac{1}{2})kxy^2 - (\tfrac{1}{2})y^2 + (\tfrac{1}{2})Aky^2 - \frac{k^2y^4}{8} - \frac{B}{k}e^{kx}.$$

Substitute this potential field and (14) into (12) to give

$$(\tfrac{1}{2})A^2 + e^{kx}\left[AB - \frac{B}{k} + Bx - (\tfrac{1}{2})kBy^2 + (\tfrac{1}{2})B^2e^{kx}\right] = E$$

which is the exact solution of System (14) in the phase plane. Here the total energy E is the parameter of the family of the phase trajectories.

It is noticed that these two examples are not solved by the conventional mathematical way. After the analogy is made, through the knowledge of classical dynamics, they are solved in a straightforward manner. There are still many other force fields which make the equation of motion integrable, but they will not be considered here.

IV. THE CLASS: $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$

Conservative Force Fields of Type II

The case for which the associated force field is conservative and the integral of energy (12) and System (1) are dependent is defined as *Type II*. The following theorem is thus introduced.

Theorem II: The necessary and sufficient condition for which the associated force field is a conservative field of Type II is

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}. \tag{15}$$

Proof: If (15) holds, then, from (4), the following relation can be proved:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Hence the associated force field is conservative. The potential energy $V(x, y)$ is obtained from (13):

$$V(x, y) = -\tfrac{1}{2}[p^2(x, y) + q^2(x, y)] + V_0, \tag{16}$$

where

$$V_0 = \tfrac{1}{2}[p^2(x_0, y_0) + q^2(x_0, y_0)] \tag{17}$$

and (x_0, y_0) is any arbitrarily chosen reference point. The quantity V_0 is defined as *reference potential* which is constant for any chosen reference point. If (16) is substituted into (12), the integral of energy becomes

$$\tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - \tfrac{1}{2}(p^2 + q^2) + V_0 = E. \tag{18}$$

The necessary condition for which the solution of these simultaneous Eqs. 1 and 18 exists is

$$E = V_0. \tag{19}$$

Under this condition, System (1) and the integral of energy (18) are dependent equations.

The reverse part is proved by use of the relationships

$$\begin{aligned} P(x, y) &= -\frac{\partial}{\partial x} V(x, y) \\ Q(x, y) &= -\frac{\partial}{\partial y} V(x, y). \end{aligned} \tag{20}$$

Since Eqs. 1 and 12 are dependent, when $\dot{x}$ and $\dot{y}$ are eliminated between them, $V(x, y)$ has to satisfy the equation

$$\tfrac{1}{2}(p^2 + q^2) + V(x, y) = E$$

identically. If (20) is applied, the same result is obtained from both of these equations, namely, condition (15). This completes the proof.

It is noted that Eq. 19 gives the condition that the particle moves

with the *same* constant energy, that is, V_0, along *any* trajectory in the domain where (15) is satisfied. This is the case because the initial condition is pre-determined by System (1).

Since the integral of energy and System (1) are dependent equations, the trajectories can not be obtained by a simple substitution as in the case of Type I. Other methods should be further developed. One of them has been completed and is given in the following paragraphs.

The P-Traj

The family of trajectories which are orthogonal to the family of phase trajectories of System (1) is called the *perpendicular trajectories*, abbreviated as "P-traj." In contrast to the P-traj, the trajectories of System (1) are called the *actual trajectories*.

The differential equation of P-traj is given as

$$\frac{dy}{dx} = -\frac{p(x, y)}{q(x, y)}, \tag{21}$$

which is the same type of equation as (2). Hence the solution of (21) exists. Furthermore, if p and q satisfy (15), P-traj can be obtained by a simple integration.

Theorem III: Assume that condition (15), $\partial p/\partial y = \partial q/\partial x$, is satisfied. Then the P-traj is given by

$$U(x, y) = \int_{x_0}^{x} p(x, y_0)dx + \int_{y_0}^{y} q(x, y)dy = c \tag{22}$$

where c is the parameter of the P-traj.

Proof: Equation 15 is equivalent to Curl $\mathbf{M} = 0$. Hence there exists a scalar function $U(x, y)$, defined by (22), such that (3, pp. 279–280) Grad $U(x, y) = \mathbf{M}(x, y)$. Thus the direction of $\mathbf{M}$ is perpendicular to the constant U-curve. Since the direction of $\mathbf{M}$ is the same direction as the actual trajectories, the actual trajectories are also perpendicular to the constant U-curve. This completes the proof.

After the P-traj is obtained, the actual trajectories can be obtained by tracing a path normal to the P-traj. Hence the actual trajectories are solved graphically.

Example 3: Consider the system

$$\begin{aligned} \dot{x} &= Ay + g(x) \\ \dot{y} &= Ax + h(y) \end{aligned} \tag{23}$$

where A is any constant. This system may represent a two-loop electrical circuit coupled by a mutual resistance. Both loops have identical self-inductances but different arbitrary nonlinear resistances as shown in Fig. 1. These nonlinear resistances could be tubes, transistors or

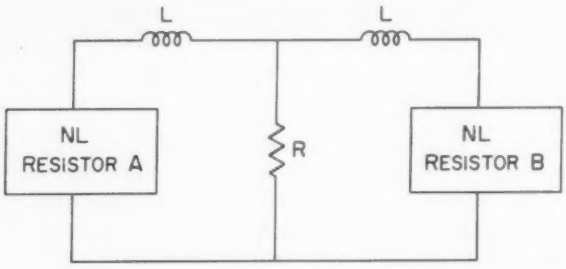

FIG. 1. Two-mesh network with nonlinear resistors to illustrate Eq. 23.

other devices. The variables x and y here are the two loop currents i_1 and i_2.

It is seen that System (23) satisfies (15). Hence, the associated force field is a conservative field of Type II. The P-traj is obtained from (22):

$$U(x, y) = A(xy - x_0y_0) + \int_{x_0}^{x} g(x)dx + \int_{y_0}^{y} h(y)dy = c \qquad (24)$$

which can be plotted if g and h are given.

Consider the simple case: $A = 1$, $g(x) = 0$, and $h(y) = (\frac{1}{2})y^2$. System (23) becomes

$$\ddot{x} - (\tfrac{1}{2})\dot{x}^2 - x = 0. \qquad (25)$$

The P-traj of (25), for $(x_0, y_0) = (0, 0)$, is thus

$$U(x, y) = xy + \frac{(y^3)}{6} = c$$

which is plotted with the actual trajectories in Fig. 2.

It is noticed, from the previous example, that even though the original system is not conservative, the associated force field may be conservative.

It can be concluded that if the associated force field is conservative, then the trajectories can be obtained either (A) analytically by a simple substitution in the case of Type I, or (B) graphically by means of P-traj in the case of Type II.

V. THE CONVERSION OF NONCONSERVATIVE FIELDS TO CONSERVATIVE FIELDS

Converting Factors

System (1) in general is considered in this section.

Multiplication of the right side of System (1) by $T(x, y)$ gives the new system

$$\begin{aligned} \dot{x} &= p(x, y)T(x, y) \\ \dot{y} &= q(x, y)T(x, y). \end{aligned} \tag{26}$$

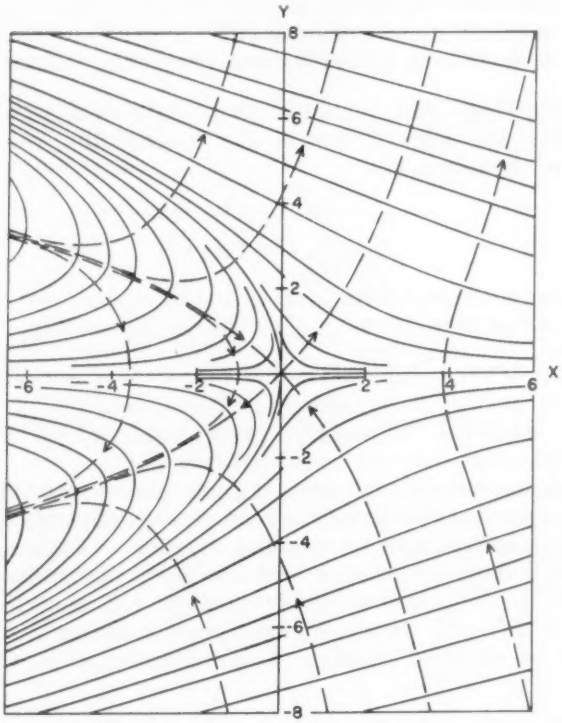

FIG. 2. The actual trajectories (dotted lines) and the perpendicular trajectories (solid lines) of the equation $\ddot{x} - \frac{1}{3}\dot{x}^2 - x = 0$ corresponding to the network of Fig. 1.

The differential equation of trajectories of the new system is

$$\frac{dy}{dx} = \frac{q(x, y)}{p(x, y)}$$

which is the same as that of the original System (1). These two systems give the same trajectories. Hence, as far as trajectories are concerned, System (1) can be replaced by the new System (26).

If $T(x, y)$ is chosen such that the relation

$$\frac{\partial}{\partial y} p(x, y)T(x, y) = \frac{\partial}{\partial x} q(x, y)T(x, y) \tag{27}$$

holds, then System (1) is converted to a new System (26) of which the associated force field is a conservative one of Type II. The function $T(x, y)$ which satisfies (27) is called the *converting factor*.

Since (27) is the partial differential equation of integrating factors (4) of the differential equation, $q(x, y)dx + p(x, y)dy = 0$, and, since there exist infinitely many integrating factors for this type of differential equation, there exist infinitely many converting factors.

Theorem IV: There exist infinitely many converting factors $T(x, y)$. Consequently, the original System (1), theoretically, can always be converted to the new system

$$\begin{aligned} \dot{x} &= p(x, y)T(x, y) \\ \dot{y} &= q(x, y)T(x, y) \end{aligned}$$

of which the associated force field is a conservative one of Type II. Hence the actual trajectories can be obtained graphically.

REMARK: The multiplication by $T(x, y)$ may introduce discontinuities, at points (x, y) which are discontinuities of $T(x, y)$ or extraneous solutions (3, p. 441): curves along which $T(x, y), = 0$. These complications are not considered here.

Now, the problem of converting a given system to a new system, of which the associated force field is conservative, depends upon finding one of the infinitely many converting factors. This is essentially a problem of finding *any* solution which satisfies the partial differential equation (27). This is theoretically possible, but, generally speaking, it is rather difficult. However, for certain p and q, the converting factor can be obtained readily. This is illustrated by the following examples.

Example 4: Consider the generalized Van der Pol equation

$$\ddot{y} + f(y)\dot{y} + g(y) = 0. \tag{28}$$

Let $F(y) = \int_0^y f(y)dy$ and $x = \dot{y} + F(y)$. System (28) can be transformed to the following expression:

$$\begin{aligned} \dot{x} &= -g(y) \\ \dot{y} &= x - F(y). \end{aligned} \tag{29}$$

This x–y plane is called the Liénard plane (5).

The converting factor of System (29) can be obtained by solving (27). Assume that $T(x, y)$ is a function of y only, then (27) can be simplified to the form

$$-\frac{d}{dy}\,g(y)T(y) = T(y)$$

which can be solved to give

$$T(y) = \frac{G(y)}{g(y)} \tag{30}$$

where $G(y) = \text{Exp}\left(-\int^{y}\frac{1}{g}\,dy\right)$. Consequently, the associated force field of the new system

$$\begin{aligned}\dot{x} &= -G(y)\\ \dot{y} &= [x - F(y)]\frac{G(y)}{g(y)}\end{aligned} \tag{31}$$

is a conservative field of Type II. The P-traj is obtained from (22):

$$-xG(y) - \int^{y}\left[F(y)\frac{G(y)}{g(y)}\right]dy = h \tag{32}$$

where h is a constant which depends on x_0 and y_0. Equation 32 is the P-traj of the generalized Van der Pol equation in the Liénard plane. For given f and g, P-traj can be plotted, and hence the trajectories can be solved graphically.

Example 5: Consider the system

$$\ddot{x} + F(\dot{x}) + x = 0. \tag{33}$$

This equation could be an electrical circuit consisting of a nonlinear resistance in series with a linear inductance and a linear capacitance, or it could be a mechanical system with a nonlinear damping and a linear spring force.

It is seen that (33) is a special case of System (29) with $g(y) = y$. Hence the converting factor can be obtained from (30): $T(y) = 1/y^2$. The P-traj is obtained from (32):

$$-\frac{x}{y} - \int^{y}\left[\frac{F(y)}{y^2}\right]dy = h. \tag{34}$$

The trajectories of System (32) thus can be obtained graphically by the use of the P-traj. In the following example, the procedure has been carried out.

The Van der Pol equation. Consider the Van der Pol equation

$$\ddot{y} - a(1 - y^2)\dot{y} + y = 0, \tag{35}$$

which is a special case of (28) with $f(y) = -a(1 - y^2)$ and $g(y) = y$.

The P-traj is found from (32) with $F(y) = -a\left(y - \frac{y^3}{3}\right)$:

$$-\frac{x}{y} + a\left(\ln|y| - \frac{y^2}{6}\right) = k \tag{36}$$

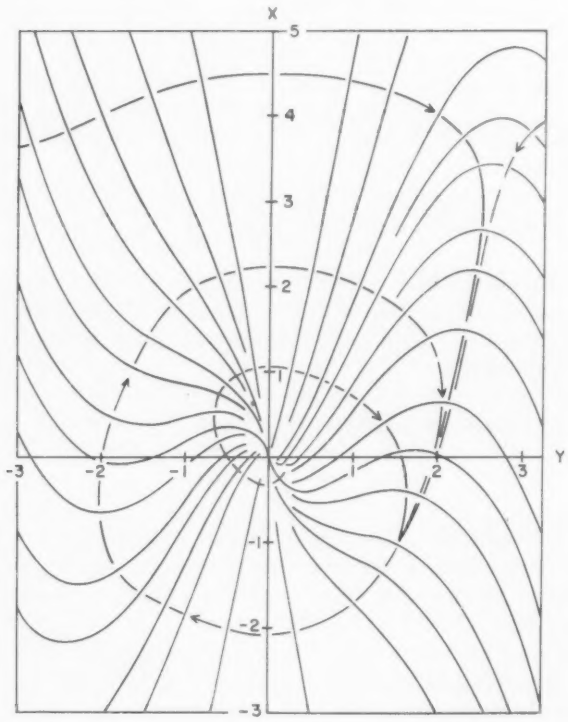

FIG. 3. The actual trajectories (dotted lines) and the perpendicular trajectories (solid lines) of the Van der Pol equation (35) with $a = 1$.

where k is a constant which includes x_0 and y_0. This is the P-traj of the Van der Pol equation in the Liénard plane. For $a = 1$, the P-traj and actual trajectories are plotted in Fig. 3.

It is seen from Fig. 3 that every actual trajectory eventually approaches a closed trajectory. This closed trajectory is called a limit cycle. The limit cycle corresponds to a steady state oscillatory solution in the actual system. Other trajectories which approach the limit cycle are the transient solutions.

There are two points which are noticed in Fig. 3 and which may be further investigated:

1. It seems that all the trajectories outside the limit cycle approach the maximum curvature curve of the P-traj first. Then they are led to the limit cycle along this maximum curvature curve. This fact may indicate some information about the transient behavior of this system.
2. It is seen from (36) that the P-traj varies continuously with the parameter a. Hence, with a varying, the transition of the P-traj can be readily found. The transition of the actual trajectories thus can be studied through the orthogonal relationship.

VI. CONCLUSIONS

Summary

An analogy from a mathematical problem to a problem of classical dynamics is developed in this paper.

After the analogy is made, three results are given:

1. If the associated equation of motion of System (1) admits an integral in the form

$$f(\dot{x}, \dot{y}; x, y; c) = 0$$

and if the integral is independent of System (1), then the phase trajectories of System (1) can be obtained by a simple substitution. Unfortunately, only a few p and q belong to this class.

2. If the relation (15), $\partial p/\partial y = \partial q/\partial x$, is satisfied, then ($a$) the associated force field is conservative and (b) the trajectories can be obtained graphically by use of P-traj.

3. If relation (15) is not satisfied, that is, considering p and q in general, then, theoretically, there exists a converting factor which converts System (1) to a new system which satisfies the relation (15). Hence results given above again will be obtained. The converting factor of the generalized Van der Pol equation has been found. The P-traj and the actual trajectories of the Van der Pol equation are plotted and discussed.

Advantages of the P-traj Method

There are three advantageous points about the method of P-traj.

1. If the P-traj is found, then the trajectories can be obtained graphically in an easy and accurate manner.

2. Because of the orthogonal relationship between the P-traj and the actual trajectories, the qualitative behavior of the latter could be analyzed by studying the curvatures and the relative positions of the former as suggested in the example of the Van der Pol equation.

3. It seems possible that the problem of finding the proper functions f, g, h or F of Systems (23), (28) and (33) to give prescribed trajectories can be solved by means of the P-traj. If the actual trajectories are given, the required P-traj can be obtained through the orthogonal relationship. It is seen from (24), (32) and (34) that the P-traj are explicit functions of f, g, h and F. Hence the most proper functions f, g, h and F could be found to give the P-traj which is closest to the required P-traj.

Further Suggestion

It is noticed from Theorem IV that every System (1) can be converted to a new system of which the force field is conservative. However, the properties of the conservative field have not been used extensively in this paper. It might be more profitable if the knowledge concerning the conservative field, such as the Hamiltonian Principle, the contact-transformations, the least action principle, etc., is further used to solve the problem. Conversely, some knowledge about the trajectories in the phase plane may lead to the knowledge about the orbits of a particle in a conservative field.

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READER OPINION

ADDENDUM TO "PONTRIAGIN'S MAXIMUM PRINCIPLE AND THE PRINCIPLE OF OPTIMALITY" *

BY

C. A. DESOER [1]

The purpose of this note is to show that Eqs. 9 of the paper reduce in all cases to the form

$$\frac{dg_i}{dt} = -\sum_{j=0}^{n} g_j \frac{\partial f_j(\mathbf{x}, \mathbf{u}^0(\mathbf{x}))}{\partial x_i}. \tag{10}$$

It has been shown in the paper that if the optimal control $\mathbf{u}^0(\mathbf{x})$ is an interior point of the set Ω, then Eqs. 9 reduce to Eq. 10. Consider now the case where for all states in some small neighborhood $N(\mathbf{x})$ of the state $\mathbf{x}$, the optimal control $\mathbf{u}^0$ is a boundary point of Ω. For convenience, let

$$\phi(\mathbf{x}, \mathbf{u}) = \langle \tilde{\mathbf{g}}(\mathbf{x}), \tilde{\mathbf{f}}(\mathbf{x}, \mathbf{u}) \rangle = \sum_{j=0}^{n} g_j(\mathbf{x}) f_j(\mathbf{x}, \mathbf{u}).$$

Since $\mathbf{u}^0(\mathbf{x})$ is a boundary point of Ω and since $\mathbf{u}^0(\mathbf{x})$ is the point at which $\phi(\mathbf{x}, \mathbf{u})$ takes its maximum value (over the set Ω), it follows that $\nabla_u \phi(\mathbf{x}, \mathbf{u})$, the gradient of $\phi(\mathbf{x}, \mathbf{u})$ with respect to $\mathbf{u}$, is parallel to the outward normal of Ω at $\mathbf{u}^0(\mathbf{x})$. Consider now the point $\mathbf{x} + d\mathbf{x}_i$ where $d\mathbf{x}_i$ is parallel to the i^{th} axis of coordinates and sufficiently small so that $\mathbf{x} + d\mathbf{x}_i \epsilon N(\mathbf{x})$. Consequently, $\mathbf{u}^0(\mathbf{x} + d\mathbf{x}_i)$ is on the boundary of Ω and $\mathbf{u}^0(\mathbf{x} + d\mathbf{x}_i) - \mathbf{u}^0(\mathbf{x}) \propto \partial \mathbf{u}^0 / \partial x_i$ is parallel to the plane tangent to the boundary of Ω at $\mathbf{u}^0(\mathbf{x})$. Therefore, the vector $\partial \mathbf{u}^0 / \partial x_i$ is orthogonal to the normal at $\mathbf{u}^0(\mathbf{x})$ and to $\nabla_u \phi$ evaluated at $(\mathbf{x}, \mathbf{u}^0(\mathbf{x}))$; hence, its scalar product is zero, *viz.*

$$\sum_{l=1}^{r} \left(\sum_{j=0}^{n} g_j \frac{\partial f_j}{\partial u_l} \right) \frac{\partial u_l{}^0}{\partial x_i} = 0. \qquad (i = 1, 2, \cdots, n)$$

Therefore, Eqs. 9 reduce to Eq. 10.

This geometric reasoning is an interpretation of a similar conclusion reached through variational techniques by Dr. L. Berkovitz in a Rand Corporation report P-2306 dated May 12, 1961.

* JOUR. FRANKLIN INST., Vol. 271, pp. 361–367 (1961).

[1] Department of Electrical Engineering, University of California, Berkeley, Calif.

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THE FRANKLIN INSTITUTE LABORATORIES
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MACROSCOPIC FERROMAGNETIC RESONANCE EQUATION FOR ARBITRARY MAGNETIC FIELD *

BY

C. J. LIN[1]

ABSTRACT

Most ferromagnetic resonance experiments are arranged in such a way that the Kittel resonance equation can be applied in its simplest form. The physical implication of this requires that: (i) the externally applied static magnetic field $\mathbf{H}_d$ be large enough that the magnetization $\mathbf{M}$ of the sample be parallel to the applied field; (ii) the applied field and the sample magnetization be along one of the principal directions of the anisotropy energy; and (iii) small angle precessions of $\mathbf{M}$ about the static equilibrium direction be assumed. Occasionally, cases are encountered which do not allow such a simplification. The present work indicates that the equation for the more general case, still assuming small angle precessions, can be derived in a manageable form by an orthogonal transformation in real space. The various energy terms involved in the transformed equation are then treated separately.

I. INTRODUCTION

A resonance equation describes the favorable condition for resonance absorption. Unless higher mode spin-wave excitation or high damping phenomena are considered, small oscillations and negligible damping torques are invariably assumed in the derivation of a resonance equation for the fundamental mode. The usual procedure assumes the magnetization vector $\mathbf{M}$ to have sinusoidal components $M_\xi' \exp(i\omega t)$ and $M_\eta' \exp(i\omega t - i\pi/2)$ along the mutually perpendicular directions ξ and η, respectively, these directions being perpendicular to the static equilibrium direction ζ, (Fig. 1). Substitution of these components in the equation

$$\dot{\mathbf{M}} = \gamma \mathbf{M} \times \mathbf{H} \tag{1}$$

yields a pair of linear homogeneous equations in M_ξ' and M_η', where γ is the magnetomechanical ratio and $\mathbf{H}$ is the sum of the externally applied static magnetic field $\mathbf{H}_s$, the demagnetization field $\mathbf{H}_d$ and the anisotropy field $\mathbf{H}_a$. From the secular equation, one obtains the precessional frequency ω in terms of $\mathbf{H}_s$ and other parameters. This is the original Kittel (1)[2] equation for complete line-up (that is, parallelism of $\mathbf{M}$, $\mathbf{H}_s$, $\mathbf{H}_d$, $\mathbf{H}_a$ is implied).

For less restricted conditions, the resulting pair of equations may not be homogeneous in M_ξ' and M_η'. Under these circumstances, the

* This work has been supported by Wright Air Development Division under Contract No. AF(616)-7499.

[1] Magnetics and Semiconductors Laboratory.

[2] The boldface numbers in parentheses refer to the references appended to this note.

following alternative set of equations of motion can be used,

$$\dot{M}_\xi = \gamma \frac{\partial E}{\partial \phi_\eta} \qquad \dot{M}_\eta = -\gamma \frac{\partial E}{\partial \phi_\xi}, \tag{2}$$

where E is the total free energy and ϕ_ξ, ϕ_η are the projections of the precessional angle on the $\zeta\xi$ and $\eta\zeta$ planes, respectively. A Taylor expansion of E in terms of ϕ_ξ, ϕ_η around the static equilibrium direction yields a pair of first order homogeneous equations of which the secular equation is (2)

$$\frac{\omega^2}{\gamma^2} = \frac{1}{M^2}\left[\frac{\partial^2 E}{\partial \phi_\xi{}^2}\frac{\partial^2 E}{\partial \phi_\eta{}^2} - \left(\frac{\partial^2 E}{\partial \phi_\xi \partial \phi_\eta}\right)^2\right]. \tag{3}$$

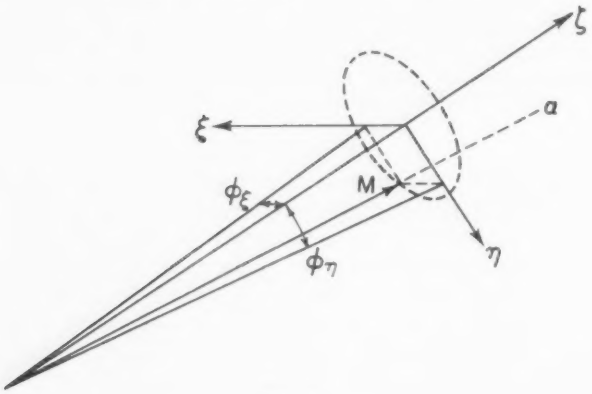

FIG. 1.

In using this expansion and neglecting higher terms, it is tacitly assumed that the amplitudes of ϕ_ξ and ϕ_η are approximately the same; hence, it is not, in general, correct to wite

$$\frac{\omega^2}{\gamma^2} = \frac{1}{M^2 \sin\theta}\left[\frac{\partial^2 E}{\partial \theta^2}\frac{\partial^2 E}{\partial \phi^2} - \left(\frac{\partial^2 E}{\partial \phi \partial \theta}\right)^2\right], \tag{4}$$

unless the pole of the spherical polar coordinates is nearly perpendicular to **M**.

Extension of the Kittel equation of all directions of **M** within 100 or 110 planes of cubic crystalline anisotropy was done by Bickford (3). More general expressions for arbitrary **M** direction were derived quantum mechanically by Van Vleck, using a quadrupole-quadrupole term in the Hamiltonian (4). They were also derived classically by MacDonald, using a variation method, for **M**, $\mathbf{H}_s$ line-up condition (5); that is, $\mathbf{H}_s$ is assumed to be so large that **M** is practically parallel to $\mathbf{H}_s$.

For most experiments, 3-cm. band frequencies are used. The H_s for free electron spin resonance is approximately 3000 oersted, while H_a can be of an order of 10^3 if M $\sim 10^2$ and $2K \sim 10^5$. Hence, the angle between **M** and $\mathbf{H}_s$ may be considerable and M may not even be able to point close to the vicinity of hard directions.

The following discussion is a way of handling the more general situation without imposing severe limitations or applying drastic assumptions.

II. MORE GENERAL APPROACH

Let $\boldsymbol{\alpha}$ be the unit vector along **M** and α_i its components or the direction cosines. To find the static equilibrium direction $\boldsymbol{\zeta}$ of $\boldsymbol{\alpha}$, it may be stated that

$$\delta E = (\nabla_\alpha E)\cdot\delta\boldsymbol{\alpha} = 0 \tag{5}$$

subject to the constraint

$$\delta\alpha^2 = 2\boldsymbol{\alpha}\cdot\delta\boldsymbol{\alpha} = 0. \tag{6}$$

Letting λ be the Lagrangian multipler, from (5) and (6) one obtains

$$(\nabla_\alpha E - \lambda\boldsymbol{\alpha})\cdot\delta\boldsymbol{\alpha} = 0. \tag{7}$$

Thus

$$\frac{\partial E}{\partial\alpha_i} = \lambda\alpha_i. \tag{8}$$

For example, on differentiating the energy in the case of cubic crystalline anisotropy, Eq. 8 yields

$$MH_s\beta_i + M^2N_i\alpha_i - 2(K_1 + K_2)\alpha_i^3 + 2K_2\alpha_i^5 = -\lambda\alpha_i \tag{9}$$

where β_i are the direction cosines of $\mathbf{H}_s$ and N_i the demagnetization factors.

Equation 9 multiplied by α_i and summed over i is

$$MH_s\sum_i\alpha_i\beta_i + M^2\sum_i N_i\alpha_i - 2(K_1 + K_2)\sum_i\alpha_i^4 + 2K_2\sum_i\alpha_i^6 = -\lambda \tag{10}$$

where β_i are the direction cosines of $\mathbf{H}_s$. By observing the structure of Eqs. 9 and 10, α_i can be solved for given β_i either exactly or more conveniently by a few cycles of iteration, that is, the static problem can be considered as a separate one whose numerical solution can be found as accurately as desired.

The principal crystalline axes x, y, z, are generally used as the coordi-

nate system for expressing the anisotropy energy, and consequently the total energy E of the system will be referred to these axes. The essential step is to transform this sytem into a primed one x', y', z', which is the same as ξ, η, ζ. This can be done by two consecutive rotations, one bringing z into ζ and the other x, y into ξ, η. The orthogonal transformation matrix can easily be written as

$$(A_{ij}) = \begin{vmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} \dfrac{\alpha_2}{\sqrt{(1-\alpha_3^2)}} & \dfrac{-\alpha_1}{\sqrt{(1-\alpha_3^2)}} & 0 \\ \dfrac{\alpha_1\alpha_3}{\sqrt{(1-\alpha_3^2)}} & \dfrac{\alpha_2\alpha_3}{\sqrt{(1-\alpha_3^2)}} & -\sqrt{(1-\alpha_3^2)} \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{c\alpha_2 + s\alpha_1\alpha_3}{\sqrt{(1-\alpha_3^2)}} & \dfrac{-c\alpha_1 + s\alpha_2\alpha_3}{\sqrt{(1-\alpha_3^2)}} & -s\sqrt{(1-\alpha_3^2)} \\ \dfrac{-s\alpha_2 + c\alpha_1\alpha_3}{\sqrt{(1-\alpha_3^2)}} & \dfrac{s\alpha_1 + c\alpha_2\alpha_3}{\sqrt{(1-\alpha_3^2)}} & -c\sqrt{(1-\alpha_3^2)} \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix} \tag{11}$$

where c and s are the cosine and sine of an angle yet to be determined. The direction cosines of **M** in the primed system are

$$\begin{aligned} \alpha_1' &= \sin\phi_\xi \\ \alpha_2' &= \sin\phi_\eta \\ \alpha_3' &= [1 - (\sin^2\phi_\xi + \sin^2\phi_\eta)]^{1/2}. \end{aligned} \tag{12}$$

The first and second derivatives of α_i' with respect to ϕ_ξ, ϕ_η are easily evaluated, and at the static equilibrium direction all those derivatives are zero except $\dfrac{\partial\alpha_1'}{\partial\phi_\xi} = \dfrac{\partial\alpha_2'}{\partial\phi_\eta} = 1$ and $\dfrac{\partial^2\alpha_3'}{\partial\phi_\xi^2} = \dfrac{\partial^2\alpha_3'}{\partial\phi_\eta^2} = -1$. Hence, applying α_i as the intermediate variables for the differentiation, Eq. 3 takes the new form as the resonance equation:

$$\frac{\omega^2}{\gamma^2} = \frac{1}{M^2}\left[\left(\frac{\partial^2 E}{\partial\alpha_1'^2} - \frac{\partial E}{\partial\alpha_3'}\right)\left(\frac{\partial^2 E}{\partial\alpha_2'^2} - \frac{\partial E}{\partial\alpha_3'}\right) - \left(\frac{\partial^2 E}{\partial\alpha_1'\partial\alpha_2'}\right)^2\right]. \tag{13}$$

Since ξ and η are presumably the principal directions of oscillation we must have

$$\frac{\partial^2 E}{\partial\alpha_1'\partial\alpha_2'} = 0. \tag{14}$$

When the terms of (14) are written according to the transformation (11), there results an equation which serves to determine s (and hence also $c = \sqrt{1 - s^2}$). In MacDonald's derivation the internal fields obtained by a variation method were put into the form of effective demagnetization fields. Then, by dropping the off-diagonal elements of the demagnetization tensor, the resonance equation was obtained in the Kittel form. The present transformation has a similar effect, but it is achieved exactly and more directly.

By means of (11), calculate the right-hand side of (13), term by term, for the standard forms of the free energy (6).

(1) Magnetostatic Energy, ${}_sE$

$$ {}_sE = - MH_s \sum_i \alpha_i\beta_i = - MH_s \sum_i \alpha_i'\beta_i' \tag{15} $$

where β_i are the direction cosines of $\mathbf{H}_s$. Let $E''_{i'j'}$ be the abbreviation for $\dfrac{\partial^2 E}{\partial\alpha_i'\partial\alpha_j'}$ and $E'_{i'}$ for $\dfrac{\partial E}{\partial\alpha_i'}$ $(i, j = 1, 2, 3)$. Then

$$ {}_sE'_{3'} = - MH_s\beta_3' = - MH_s \sum_i \alpha_i\beta_i \tag{16} $$

$$ {}_sE''_{1'1'} = 0. \tag{17} $$

(2) Demagnetization energy ${}_dE$

$$ {}_dE = - \frac{M^2}{2} \sum_i \alpha_i^2 N_i \tag{18} $$

where N_i are the demagnetization factors (assuming the principal axes of the ellipsoid to be the same as the crystalline axes). Then

$$ {}_dE'_{3'} = - M^2 \sum_i A_{3i}^2 N_i = - M^2 \sum_i \alpha_i^2 N_i \tag{19} $$

$$ \begin{aligned} {}_dE''_{1'1'} = - M^2 \sum_i A_{1i}^2 N_i = - M^2[&(c\alpha_2 + s\alpha_1\alpha_2)^2 N_1 \\ &+ (- c\alpha_1 + s\alpha_2\alpha_3)^2 N_2 - s^2(1 - \alpha_3^2)^2 N_3]/(1 - \alpha_3^2) \end{aligned} \tag{20} $$

$$ \begin{aligned} {}_dE''_{2'2'} = - M^2 \sum_i A_{2i}^2 N_i = - M^2[&(- s\alpha_2 + c\alpha_1\alpha_3)^2 N_1 \\ &+ (s\alpha_1 + c\alpha_2\alpha_3)^2 N_2 - c^2(1 - \alpha_3^2)^2 N_3]/(1 - \alpha_3^2) \end{aligned} \tag{21} $$

$$ \begin{aligned} {}_dE''_{1'2'} = - M^2 \sum_i A_{1i}A_{2i} N_i = - M^2[&sc(N_3 - N_1)(\alpha_2^2 - \alpha_3^2\alpha_1^2) \\ &+ (s^2 - c^2)\alpha_1\alpha_2\alpha_3(N_2 - N_1)]/(1 - \alpha_3^2). \end{aligned} \tag{22} $$

(3) Cubic Crystalline Anisotropy ${}_cE$

$$ {}_cE = L_0 + L_1 \sum_i \alpha_i^4 + L_2 \sum_i \alpha_i^6 \tag{23} $$

where

$$ \begin{aligned} L_0 &= \frac{K_1}{2} + \frac{K_2}{6} \\ L_1 &= -(K_1 + K_2)/2 \\ L_3 &= K_2/3. \end{aligned} \tag{24} $$

Let ${}_{c1}E = L_1 \sum_i \alpha_1^4$ and ${}_{c2}E = L_2 \sum_i \alpha_i^6$ be calculated separately.

$$ {}_{c1}E'_{3'} = 4L_1 \sum_i A_{3i}^4 = 4L_1 \sum_i \alpha_i^4 \tag{25} $$

$$ \begin{aligned} {}_{c1}E''_{1'1'} &= 12L_1 \sum_i A_{3i}^2 A_{1i}^2 \\ &= 12L_1[c^2\alpha_1^2\alpha_2^2 + cs(\alpha_1^2 - \alpha_2^2)\alpha_1\alpha_2\alpha_3 \\ &\qquad + s^2\alpha_3^2(\alpha_1^4 + \alpha_1^2\alpha_2^2 + \alpha_2^4)]/(1 - \alpha_3^2) \end{aligned} \tag{26} $$

$$ \begin{aligned} {}_{c1}E''_{2'2'} &= 12L_1 \sum_i A_{3i}^2 A_{2i}^2 \\ &= 12L_1[s^2\alpha_1^2\alpha_2^2 - cs(\alpha_1^2 - \alpha_2^2)\alpha_1\alpha_2\alpha_3 \\ &\qquad + c^2\alpha_3^2(\alpha_1^4 + \alpha_1^2\alpha_2^2 + \alpha_2^4)]/(1 - \alpha_3^2) \end{aligned} \tag{27} $$

$$ \begin{aligned} {}_{c1}E''_{1'2'} &= 12L_1 \sum_i A_{3i}^2 A_{1i} A_{2i} \\ &= 12L_1[-2sc\alpha_1^2\alpha_2^2 - (s^2 - c^2)(\alpha_1^2 - \alpha_2^2)\alpha_1\alpha_2\alpha_3 \\ &\qquad + 2sc\alpha_3^2(\alpha_1^4 + \alpha_2^2\alpha_3^2 + \alpha_2^4)]/(1 - \alpha_3^2) \end{aligned} \tag{28} $$

$$ {}_{c2}E'_{3'} = 6L_2 \sum_i A_{3i}^6 = 6L_2 \sum_i \alpha_i^6 \tag{29} $$

$$ \begin{aligned} {}_{c2}E''_{1'1'} &= 30L_2 \sum_i A_{3i}^4 A_{1i}^2 \\ &= 30L_2[c^2\alpha_1^2\alpha_2^2 + 2sc\alpha_1\alpha_2\alpha_3(\alpha_1^2 - \alpha_2^2) + s^2\alpha_3^2] \end{aligned} \tag{30} $$

$$ \begin{aligned} {}_{c2}E''_{2'2'} &= 30L_2 \sum_i A_{3i}^4 A_{2i}^2 \\ &= 30L_2[s^2\alpha_1^2\alpha_2^2 - 2sc\alpha_1\alpha_2\alpha_3(\alpha_1^2 - \alpha_2^2) + c^2\alpha_3^2] \end{aligned} \tag{31} $$

$$ {}_{c2}E''_{1'2'} = 30L_2[-sc\alpha_1^2\alpha_2^2 - (s^2 - c^2)\alpha_1\alpha_2\alpha_3(\alpha_1^2 - \alpha_2^2) + sc\alpha_3^2]. \tag{32} $$

The less frequently encountered energy terms, such as crystalline anisotropy of uniaxial and hexagonal symmetries and magnetostriction energy, can also be treated in the same manner.

III. APPLICATION TO SIMPLER CASES

Case 1

Consider an ellipsoidal sample with **M** and $\mathbf{H}_i$ in the [001] plane of cubic crystalline anisotropy. From Eqs. 17, 22, 28, and 32 it is obvious that (14) is satisfied if $s = 1$, $c = 0$, or $s = 0$, $c = 1$. Taking $s = 1$, then by Eqs. 14 to 32, the resonance equation (13) becomes

$$\begin{aligned}\frac{\omega^2}{\gamma^2} = \frac{1}{M^2}\,&[MH_i \sum_i \alpha_i\beta_i + M^2(\alpha_1^2N_1 + \alpha_2^2N_2) \\ &- 4L_1(\alpha_1^4 + \alpha_2^4) - 6L_2(\alpha_1^6 + \alpha_2^6) + M^2N_3 + 30L_2] \\ &\times [MH_i \sum_i \alpha_i\beta_i + M^2(\alpha_1^2N_1 + \alpha_2^2N_2) \\ &- 4L_1(\alpha_1^4 + \alpha_2^4) - 6L_2(\alpha_1^6 + \alpha_2^6) \\ &+ M^2(\alpha_2N_1 - \alpha_2N_2) + 24L_1\alpha_1^2\alpha_2^2 + 30L_2\alpha_1^2\alpha_2^2]. \quad (33)\end{aligned}$$

Let $\alpha_1 = \cos\theta$, $\alpha_2 = \cos\theta$, $\alpha_3 = 0$; then Eq. 33 reduces to exactly the result obtained by Bickford in the **M**, $\mathbf{H}_i$ aligned case.

Case 2

Spherical sample, $\mathbf{H}_i$ and **M** in 110 plane of cubic crystalline anisotropy $s = 1$ also is a solution of Eq. 14, but not if these samples are not spherical. Here $\alpha_3 = \cos\theta$, $\alpha_1 = \alpha_2 = \frac{1}{\sqrt{2}}\sqrt{1 - \cos^2\theta}$.

$$\begin{aligned}\frac{\omega^2}{\gamma^2} = \frac{1}{M^2}\Bigg[&MH_i \sum_i \alpha_i\beta_i - 4L_1 \sum_i \alpha_i^4 - 6L_2 \sum_i \alpha_i^6 \\ &+ \frac{24L_1\alpha_3^2(\alpha_1^4 + \alpha_1^2\alpha_2^2 + \alpha_2^4)}{1 - \alpha_3^2} + 30L_2\alpha_3^2\Bigg] \\ \times \Bigg[&MH_i \sum_i \alpha_i\beta_i - 4L_1 \sum_i \alpha_i^4 - 6L_2 \sum_i \alpha_i^6 \\ &+ \frac{24L_1\alpha_1^2\alpha_2^2}{1 - \alpha_3^2} - 30L_2\alpha_1^2\alpha_2^2\Bigg] \\ = \frac{1}{M^2}\,&[\mathbf{M}\cdot\mathbf{H}_i - 4L_1(\cos^4\theta + \tfrac{1}{2}\sin^4\theta) \\ &- 6L_2(\cos^6\theta + \tfrac{1}{4}\sin^6\theta) + 18L_1\sin^2\theta\cos^2\theta + 30L_2\cos^2\theta] \\ &\times [\mathbf{M}\cdot\mathbf{H}_i - 4L_1(\cos^4\theta + \tfrac{1}{2}\sin^4\theta) \\ &- 6L_2(\cos^6\theta + \tfrac{1}{4}\sin^6\theta) + 6L_1\sin^2\theta - \frac{15}{2}L_2\sin^4\theta]. \quad (34)\end{aligned}$$

The crystalline anisotropy terms of Eq. 34 can again be reduced to exactly the same as Bickford's. However, the terms $(N_1 - N_3)M^2$ and

$(N_2 - N_3)M^2$ cannot simply be added in Eq. 34 in order to include demagnetization energy for the general ellipsoidal sample since a more complicated solution than $s = 1$ is required to make ${}_dE''_{1'2'} + {}_cE''_{1'2'} = 0$.

Case 3

Spherical sample, first order cubic crystalline anisotropy, general **M** direction. The resonance equation (Eq. 13) for this case becomes

$$\left(\frac{\omega}{\gamma}\right)^2 = \frac{1}{M^2}(\mathbf{M}\cdot\mathbf{H}_s + {}_{c1}E'_{1'1'} - {}_{c1}E'_{3'})(\mathbf{M}\cdot\mathbf{H}_s + {}_{c1}E''_{2'2'} - {}_{c1}E'_{3'}). \quad (35)$$

Solving ${}_{c1}E''_{1'2'} = 0$, yields

$$s = \sin\left[\tfrac{1}{2}\tan^{-1}\frac{(\alpha_1^2 - \alpha_2^2)\alpha_1\alpha_2\alpha_3}{\alpha_1^2\alpha_2^2 - \alpha_3^2(\alpha_1^4 + \alpha_1^2\alpha_2^2 + \alpha_2^4)}\right] \quad (36)$$

and c, which equals the cosine of the same bracket. Hence ${}_{c1}E'_{3'}$, ${}_{c1}E''_{1'1'}$, ${}_{c1}E''_{2'2'}$ in Eq. 35 are given by Eq. 36 and Eqs. 25, 26, 27, respectively.

IV. DISCUSSION

In Sections II and III it was shown that cases more general than the **MH**$_s$ line-up one can be treated without much difficulty. Solving for the static equilibrium first greatly simplifies the procedure. In the MacDonald or Van Vleck theory, the direction of the microwave magnetic field was conveniently taken as the ξ axis (present notation). It's magnitude was taken as negligible and did not enter the theory in any way. Hence the ξ direction seemed to be somewhat arbitrary. In the present work ξ is fixed by Eq. 14. Actually, Eq. 13 alone will give the unique solution for the stable precession.

Quantum mechanical justification of the macroscopic result can be made by following every step of Van Vleck's derivation, except that the direction ξ is taken to be the quantization direction which is in general not the same as that of $\mathbf{H}_s$. After simplification of terms by various approximations as used by Van Vleck, there results

$$\begin{aligned} \dot{S}_\xi &= aS_\xi + b_1S_\eta + c_1 \\ \dot{S}_\eta &= b_2S_\xi - aS_\eta + c_2 \\ \dot{S}_\zeta &= w^{-1}(-c_1S_\xi - c_2S_\eta) \end{aligned} \quad (37)$$

where a, b_1, b_2, c_1, c_2, and w are functions of direction cosines. Since ζ is presumably the direction of quantization, $\dot{S}_\zeta = 0$; hence $c_1 = c_2 = 0$, By setting $a = 0$, the direction of ξ (and η) is fixed just as in Section II. hence the transformation relation is exactly the same as Eq. 11. Consequently a similar result can be expected.

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THE BLACK VOID REACTOR CONCEPT

The National Bureau of Standards has been investigating a promising concept in research reactor design—that of the Black Void Reactor[1], or BVR. The BVR can simultaneously provide both fast and slow neutron fluxes that are spatially separated. An arrangement that provides this separation, conceived by C. O. Muehlhause of the Bureau staff, would permit radiation effects studies and thermal neutron beam studies (for example, neutron diffraction) to be conducted independently in the same reactor. In addition, this arrangement constitutes a type of flux trap (inverted) which yields higher fluxes for a given power input than does the usual research reactor.

During a design study for the National Bureau of Standards Reactor (NBSR)[2], to be constructed at the new NBS site at Gaithersburg, Maryland, consideration was given to the idea of incorporating a converter assembly into the NBSR design. This consideration led to the BVR concept of reactor design. Although the BVR appears to possess a number of advantages, it requires significant fuel element development and testing before it can be applied. Two new reactor types[3] under consideration by the Atomic Energy Commission may contribute technique to the development of the heavily loaded annular fuel elements required for the BVR.

The general type of Black Void Reactor is the BVR-N, where N represents the number of heavily loaded (appearing black to thermal neutrons) cylindrical fuel assemblies in the configuration. As in other reactors, the fast neutrons released by fission in the fuel walls are slowed by the moderator, and a certain portion of these now slow, or thermal, neutrons return to the fuel assembly. Some of these thermal neutrons will be absorbed by the fuel walls and converted into fast fission neutrons to maintain the chain reaction.

In the BVR, a portion of the fast fission neutrons and some neutrons of intermediate energy that have been partially moderated irradiate the void interior of the tube. A sizeable space is thus provided in which instrumented samples may be placed for radiation effects studies.

[1] For further information, see "The Black Void Reactor Concept," by C. O. Muehlhause, *Nuclear Sci. and Eng.*, Vol. 7, p. 505 (June, 1960).

[2] "High-Flux Research Reactor for New NBS Site," *NBS Tech. News Bull.*, Vol. 45, (August, 1961).

[3] High Flux Beam Reactor at Brookhaven National Laboratory, and High Flux Isotope Reactor at Oak Ridge National Laboratory.

At the same time, beam tubes in the moderator outside the fuel assemblies can provide thermal neutrons for use in neutron diffraction studies.

The simplest approach to the Black Void Reactor is the BVR-1, consisting of a single long cylindrical fuel assembly immersed in a large tank of moderating material. A practical example would be an assembly eight feet in length and eight inches in diameter. According to the design study, fast neutrons leaving the fuel walls would slow down in the moderator and return to the walls with a probability of 0.543. The over-all multiplication of the system (number of fast fission neutrons produced per neutron absorbed in the fuel walls) must be adjusted to have the inverse of this probability value. At a power of three megawatts, a fast neutron flux of approximately $10^{14}/cm^2$ sec would be available within the cylinder. A limited number of beam tubes may be placed in the moderator of the BVR-1.

Another example of the BVR is the symmetrical BVR-3, which comprises three 6-in.-diameter, 8-ft.-long cylindrical assemblies spaced 1.73 ft. apart. Here the return probability for neutrons leaving one assembly and being absorbed in any of the three assemblies is 0.642. Thus, the walls of each assembly must have a multiplication in excess of 1.56.

The NBS design study showed that as a research reactor, the BVR should have the advantages of long assembly life (greater than 1 yr. at several megawatts) and safe operation due to long mean neutron lifetime. There is adequate excess reactivity for the thermal neutron beam tubes. The beams may be readily aimed at the space between assemblies and thus avoid virgin fission neutrons and gamma rays.

TWO RADIOACTIVITY STANDARD MATERIALS

Two additional standard samples of radionuclides—iron 55 and promethium 147—are being issued by the National Bureau of Standards. These two standards,[4] part of a group of 51 now available, result from the Bureau's continual program to establish new and improved radioactivity standards.

Both of the standards have wide applications for medical and industrial laboratory purposes. Iron 55, an electron-capturing nuclide with a half-life of 2.7 years, decays by X-ray and Auger-electron emission directly to the ground state. It is used extensively as a tracer both in biological and industrial research.

Promethium 147 is a pure beta emitter utilized for generation of

[4] Standard materials issued by the National Bureau of Standards are described in NBS Circular 552, Standard Materials. This Circular may be ordered for 35 cents from the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C. Up-to-date supplementary inserts, which are issued periodically, are available upon request directly from the National Bureau of Standards.

low-energy X-rays and energy calibration standards for X- and gamma-ray spectrometers. It has a half-life of 2.5 years. In addition to other research applications, this radionuclide is incorporated in luminescence markers and is being considered for use in nuclear batteries.

Both of the standard samples may be ordered under the general licensing provisions of the Atomic Energy Act of 1954. The iron 55 standard sample No. 4929 and the promethium 147 standard sample No. 4940 may be purchased for $36.00 and $40.00, respectively. Both samples are in solution form (approximately 3 ml), and have nominal activities of 5×10^4 dps/g. The iron 55 solution contains 1μg/ml $FeCl_3$ as carrier, while the promethium 147 solution is carrier free. All orders for the radioactivity standards should be addressed to (Miss) Elizabeth M. Zandonini, Radioactivity Standard Samples, Radioactivity Section, National Bureau of Standards, Washington 25, D. C.

BOOK REVIEWS

THEORY OF MACHINES, by Joseph E. Shigley. PART I, KINEMATIC ANALYSIS OF MECHANISMS, 631 pages, diagrams, illustrations, 6 × 9¼ in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $11.50. PART II, DYNAMIC ANALYSIS OF MACHINES, 645 pages, diagrams, illustrations, 6 × 9¼ in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $7.50.

This very complete and well-written book treats the standard as well as the more advanced topics of machine analysis with great clarity. The use of vector notation is introduced in the first chapter to lay a solid foundation for the exposition of kinematics, and of the geometry of motion, which comprise over one third of Part 1. This is followed by chapters on gearing, gear trains, linkages and cams which, in keeping with the purpose of the book, are discussed from the point of view of analysis rather than design. Many references are given to more detailed design applications. The advantages of vector methods are clearly illustrated in the chapter on space mechanisms. A brief discussion of analog computing mechanisms concludes Part 1.

Part 2 applies and adapts the more advanced concepts and techniques of dynamic analysis to a representative selection of practical problems. Vector methods are consistently used to solve dynamic problems in plane and space mechanisms. The discussion of the phase-plane delta method for analyzing transient vibration phenomena is clear and comprehensive. Considerable space is devoted to the dynamics of rotating machinery, including problems of balancing, and to cam mechanisms composed of non-rigid components. A brief introduction to the dynamics of feedback control systems concludes the book.

Comparable works have seldom satisfied, at the same time, the rigorous standards of the theoretician and the demands of the practicing engineer, for techniques applicable to a great variety of technical problems. Professor Shigley has succeeded in presenting the analytical foundations of machine design in a manner simple enough to be of value to the student, yet so comprehensive and general that it may serve as a reference to the design engineer.

KLAUS CAPPEL
The Franklin Institute Laboratories

LECTURES ON LINEAR ALGEBRA, by I. M. Gel'fand. 185 pages, 6 × 9 in. New York, Interscience Publishers, Inc., 1961. Price, $6.00.

The last few years have seen the publication of a good many books bearing the same or a very similar title. These books vary widely in their approach and their style. Many are written in such an abstract and "uncompromising" style that only professional mathematicians can read them with profit.

The merit and charm of this book lie in the rare blend of mathematical elegance, pedagogy and compactness. The author prepares the reader to the next development and motivates the extensions and definitions before stating them. He frequently appeals to intuitions and analogies, leading naturally to generalized concepts. No mathematician could question the rigor of the theorems, proofs and definitions as they are exposed, yet the reader has a feeling of participation in the build-up of the theory. Carefully chosen and very diverse illustrative examples contribute to improve the understanding and dispel possible misgivings. A surprising amount of material is covered in less than two hundred pages. Let us mention in particular: real and complex n-dimensional spaces; Euclidean spaces; reduction of quadratic forms (a very thorough treatment); linear, hermitean and unitary transformations; the eigenvalue problem (extremal properties) and an introduction to the algebra of tensors.

The book ends rather abruptly and the chapter on tensors seems unfinished. Two important appendices have been deleted (at the author's request) from the English edition,

and the publishers (in this otherwise carefully printed and bound book) have omitted an index of topics. Forgetting these minor defects, the book appears to be a very suitable introduction to this important branch of mathematics and mathematical physics.

HENRI AMAR
Temple University

FLOW OF FLUIDS THROUGH POROUS MATERIALS, by Royal E. Collins. 263 pages, diagrams, illustrations, 6 × 9 in. New York, Reinhold Publishing Corp., 1961. Price, $12.50.

In the preface of this text the author indicates the importance of flow through porous materials in many phases of petroleum engineering, soil mechanics, ground-water hydrology and sanitary engineering. This book is intended as a text for the petroleum reservoir engineer specifically. The format consists of ten chapters dealing with structure, statics, physical and mathematical theory, steady and transient laminar flow of homogeneous fluids, simultaneous flow of immiscible fluids, moving boundary problems, theory of models and flow with phase change.

The treatment of each chapter is a highly theoretical, mathematical development beyond the capabilities of the average undergraduate scientist or engineer. Every reviewer has favorite portions of technical publications and in this case it was the chapters concerned with moving boundary problems and flow with change of phase. The former should be of interest to petroleum producing engineers, deep-well water investigators and others dealing with similar problems of calculating complex drawdown patterns in multi withdrawal fields. Less obvious is the extension of the theoretic concepts to the complex drying or extraction process controlled by unsteady state parameters. Admittedly, the latter chapter is treated rather briefly, but the discussions of phase distribution and saturation will be of interest not only to the obvious audience, but to those researchers in chromatography and solvent extraction.

The many derivations are smoothly presented with adequate explanatory wording to permit the practicing researcher or student to follow without undue cross-referencing or assumptions. This is an excellent treatment of a subject that has not been too well elaborated by the scientific writers. Although it describes basic concepts applicable to many fields, the advanced and theoretical nature of the text will limit its circulation to research oriented readers.

ROBERT A. BAKER
The Franklin Institute Laboratories

THE THEORY OF TRANSITION-METAL IONS, by J. S. Griffith. 455 pages, diagrams, 7 × 10 in. New York, Cambridge University Press, 1961. Price, $17.50.

The subject matter of this excellent work is somewhat less general than the title would indicate, since it is restricted to what is called ligand-field theory. Metallic compounds and alloys of transition elements, and, of course, the pure elements themselves, are excluded. The theory applies to situations in which the states which accommodate the d- or f-orbitals of the isolated transition element ion are, to a good approximation, localized on the central transition element ion and its nearest neighbors or ligands. Cooperative phenomena involving several transition element ions are also excluded from the treatment.

After a brief introductory chapter, the author begins with the quantum mechanical description of the angular momentum of the free atom. The bra and ket notation of Dirac is used here and throughout the book. There is a short chapter on electromagnetic radiation and its interaction with an isolated atom. Chapters 4 and 5 deal with the structure of free atoms and ions and with magnetic effects, respectively. Spin is introduced first phenomenologically and then via Dirac's wave equation. Chapter 6 concludes the survey of background material with a treatment of group theory and matrix representations of groups. An effort is made to collect useful theorems and formulae for the benefit of those using the work as a reference only. These collections are unfortunately not easily found without paging through the text.

With Chapter 7 the concept of complex ion is introduced. The possible coupling schemes are discussed. These depend on the order in which the electrostatic interaction between the partially filled shells and the rest of the ion ($\mathcal{U}$), the mutual electrostatic interaction

of the electrons in the ion-filled shells ($\mathcal{H}_e$), and the spin-orbit interaction energy ($\mathcal{H}_s$) are diagonalized. If the order is $\mathcal{H}_e$, $\mathcal{H}_s$, $\mathcal{V}$, the treatment is equivalent to crystal field theory. This order is termed the rare earth coupling scheme for obvious reasons. In the weak field coupling scheme the order is $\mathcal{H}_e$, $\mathcal{V}$, $\mathcal{H}_s$, and in the strong field coupling scheme it is $\mathcal{V}$, $\mathcal{H}_e$, $\mathcal{H}_s$. The first two schemes are treated in Chapter 8 and the third in Chapter 9. In Chapter 10 the paramagnetic susceptibility of different electronic configurations are calculated, and it is shown how the experimental determination of susceptibility often leads to an unambiguous spin assignment to the electronic configuration of the complex ion. Optical spectra and thermodynamic properties of some complex ions are calculated and compared with experimental results in Chapter 11. Chapter 12 is devoted to the calculation of the electron and nuclear spin resonance spectra of the complex ions. Appendices are provided for special topics not treated in the text.

The book is well written and referenced. The explanations are clear and the style, after the first shock of meeting the first person singular, stimulating. Exercises are provided after most of the sections. This is a valuable textbook and useful reference.

F. J. DONAHOE
The Franklin Institute Laboratories

A HOLE IN THE BOTTOM OF THE SEA, by Willard Bascom. 352 pages, diagrams, illustrations, plates, $5\frac{1}{4} \times 9\frac{1}{4}$ in. New York, Doubleday & Co., Inc., 1961. Price, $4.95.

The statement that we know more about the distant stars than we do about the earth on which we live is an obvious truth. Analytical boring machines have yielded a fairly comprehensive picture of the inside of a star. Not to be outdone, the earth scientists are trying to unlock the secrets of the earth to provide more knowledge of our planet. They have conjured up a fantastic scheme proposed by as unlikely a society as can be imagined and the combination of the two has resulted in this country being involved in the drilling of a hole in the bottom of the sea.

For a long time, seismologists have tried to determine the structure of the earth. Shrewd guesses and skillful interpretations of seismic waves have yielded some knowledge of the earth's structure. In the interpretations, scientists discovered a discontinuity which comes within 6 miles of the sea's surface. Because the earth is so much thicker on land, the obvious approach to pierce the surface layer is to drill the hole under the sea. Thus results the attempt to bore through the upper crust of the earth to uncover the composition of the subcrustal layers. There is a bonus to be taped in this process. When the drill brings up cores of the ocean sediments it may prove to be a key to millions of years of the past life history of the earth.

A Hole in the Bottom of the Sea is the story of the origin and the implementation of the drilling of the Mohole. It is the story of the scientists' attempt to breach the crustal layers of the earth to learn more about the structure. In a magnificently written book, Willard Bascom has chronicled the story of how the idea originated and what is being done and what scientists hope to do. The author tells his story in a fascinating manner and skillfully weaves a tremendous volume of information in the course of telling the story. Even the technical details in the art of drilling holes both on land and on sea prove excellent exposition. The use of line drawings proves a valuable asset in visualizing the rigs and their uses.

Despite the fact that the author is the director of the Mohole Project, he does not paint too rosy a picture concerning the success of the project. He indicates the tremendous technical difficulties which will be encountered and possible solutions. Fortunately he is a realist and recognizes fully the magnitude of the task.

Several minor errors have crept into this work when the author gets involved in astronomy. He speaks of the shift of the red calcium band. The calcium lines are in the blue. He speaks about the farthest stars moving fastest away. The stars do not move in the sense of all moving away from the earth —it is the star systems or galaxies to which the author refers. In his table of planets and moon the number of Jovian moons is given as 11—there are 12.

These are minor items in one of the most fascinating books it has been this reviewer's pleasure to read. For those who are inquisi-

tive about one of the more startling adventures of science they can do no better than to read the exciting *A Hole in the Bottom of the Sea*.

I. M. LEVITT
The Fels Planetarium

THE OPTIMAL DESIGN OF CHEMICAL REACTORS. VOL. 3, MATHEMATICS IN SCIENCE AND ENGINEERING, by Rutherford Aris. 191 pages, diagrams, 6 × 9 in. New York, Academic Press Inc., 1961. Price, $7.00.

Advanced mathematical concepts coupled up with computer solutions have become increasingly important in the solution of complex engineering problems. Dynamic programming has proven especially powerful and versatile in solution of the engineering and economic aspects of reactor design. The practicing design engineer and mathematician have had to cooperate to a greater extent than ever before, but often without complete comprehension of the other's language. Elucidation of this common language has been effectively accomplished for both by this treatise.

The author's personality comes through in a manner not common in scientific discussions of depth. Chapters are prefaced by witty rhymes, an excellent style with choice vocabulary describes the subject matter, and there are liberal cautions against building a mathematical house of cards based on unnecessary complications. The content is very mathematical as befits the subject. The reader should be grounded in such basic advanced mathematical concepts as probability distributions, transforms, etc., to follow the derivations. An understanding of chemical engineering kinetics and thermodynamics will also be useful.

Reactor designs have previously been developed through trial and error experience. As understanding has evolved, basic concepts have been developed in mathematical format. The modern reactor design problem involves many parameters and manpower and time considerations preclude the classical engineering approach. Reduction of the overall problem to manageable portions for solution by digital computers via dynamic programming has proven ideal. Since the design premises may be stated with fairly high accuracy, engineering design by dynamic methods is sounder than solution in other fields such as economics.

There are three components of optimal design. The specification of: (1) process stream state, (2) operating variables and transformations they effect, and (3) objective function to be optimized. In this book functional equations suitable for digital computer solution are evolved for a variety of reactor types. Consideration is given to physical and chemical factors with the number of independent reactions being indicative of the problem complexity. Cost estimation procedures are evolved based on (1) capital and maintenance costs (2) stream costs (3) heat costs and (4) control costs. Functions are developed for unit profit per unit time. Optimal equations are derived for continuous flow stirred reactors, multibed adiabatic reactors and tubular reactors. Temperature and holding time, feed stream bypassing, interchange and cold shot cooling, etc. are considered where applicable. Stochastic applications are given and two in particular show the need for such treatment. These are optimal replication of processes subject to failure, and optimal catalyst replacement policies. In the latter case a management policy is obtained through maximization of profit vs. catalysts replacement cost. A brief treatment of optimizing operation of existing reactors completes the technical content divided into nine chapters.

References are given at the end of the book and it is vividly clear that only a handful of authors all of current vintage are represented. This is a new, important subject. Chemists, chemical engineers, designers, operations analysts and computer specialists will benefit by this work. The publisher's cover includes undergraduates in the suggested audience. Unless the undergraduate educational processes are suddenly accelerated with computer speed this is likely to be an overenthusiastic statement. Scientists and mathematicians will find this a very valuable addition to their libraries.

R. A. BAKER
The Franklin Institute Laboratories

BOOK NOTES

REFERENCE ELECTRODES: THEORY AND PRACTICE, edited by David J. G. Ives and George J. Janz. 651 pages, diagrams, 6 × 9 in. New York, Academic Press, 1961. Price, $20.00.

This new graduate-level text is principally concerned with experimental aspects of electrochemical theory. Reference electrodes are examined from both the mechanistic and thermodynamic viewpoints. No separation is made between theory and experiment; each chapter contains both. After an introductory chapter, eight chapters deal with specific kinds of electrodes—for example, hydrogen, calomel, silver halide, quinhydrone—and the three final chapters cover general topics such as reference electrodes in nonaqueous solutions, biological uses of microelectrodes, and electrodes in fused salt systems. Not the least attractive feature of the book is the extensive list of references appended to each chapter. The volume, in addition to its usefulness in filling a gap in graduate texts, will be of interest to research workers.

LECTURES IN THEORETICAL PHYSICS, VOL. III, edited by Wesley E. Brittin, B. W. Downs and Joanne Downs. 531 pages, diagrams, 6 × 9 in. New York, Interscience Publishers, Inc., 1961. Price, $11.00.

Volume III of this series contains lectures delivered at the Summer Institute for Theoretical Physics at the University of Colorado in 1960. The level of presentation of the eleven lectures lies between that of the textbook and that of the research journal. The wide variety of subjects and the eminence of the contributors are evident from this listing of the papers: "Causality and Dispersion Relations," by Aage Bohr; "Selected Topics in Theoretical Physics," by Victor Weisskopf; "Statistical Mechanics of Irreversibility," by Robert Zwanzig; "The Scattering of Electrons by Atoms," by B. L. Moiseiwitsch; "Some Applications of the Generating Functional of the Molecular Distribution Functions," by Melville Green; "Nonequilibrium Statistical Mechanics," by Elliott Montroll; "Quantum Theory of Collision Processes," by R. Haag; "Canonical Commutation Relations in Field Theory and Functional Integration," by R. Haag; "General Theory of Nonequilibrium Phenomena," by Radu Balescu; "Martin-Schwinger Methods in the Many-Body Problem," by Neil Ashby; and "Green's Functions and the Quantum Theory of Fields," by Kurt Symanzik.

INTRODUCTION TO ELEMENTARY PARTICLE PHYSICS, by R. E. Marshak and E. C. G. Sudarshan. Interscience Tracts on Physics and Astronomy #11, 217 pages, tables, 5¼ × 8 in. New York, Interscience Publishers, 1961. Price: $2.50 (paper); $4.50 (cloth).

This tract concentrates on those aspects of elementary particle physics which do not depend on detailed dynamical calculations. The physical meaning of the various concepts is emphasized and the experimental material is introduced chiefly for illustrative purposes. The book is intended as a brief introduction for the non-expert in theoretical high energy physics.

PHYSICAL CHEMISTRY, by Gordon M. Barrow. 684 pages, diagrams, illustrations, 6¼ × 9¼ in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $8.95.

An introductory, undergraduate text, this work deals with all the subjects customarily dealt with, such as thermodynamics, states of matter and kinetics. The author unifies the various subjects by interpretations based on molecular structure and behavior, for he believes that physical chemistry is not a branch of chemistry, but rather a method which is applicable to all branches of chemistry. The text is roughly divided into two parts; the first deals with situations involving a single or a few molecules at a time, while

the second involves the interplay of many molecules.

HANDBOOK OF THERMOPHYSICAL PROPERTIES OF SOLID MATERIALS, compiled by Alexander Goldsmith, Thomas E. Waterman and Harry J. Hirschhorn. Five volumes, revised edition, 4300 pages, diagrams, charts, 8½ × 11 in. New York, The Macmillan Co., 1961. Price, $90.00 (five volumes).

Prepared by Armour Research Foundation under contract to the U. S. Air Force, this five-volume reference work provides a much needed source of reliable thermophysical property data on high temperature materials. Twelve properties are included: density, melting point, latent heat of fusion, latent heat of vaporization, latent heat of sublimation, specific heat, thermal conductivity, thermal diffusivity, emissivity and reflectivity, linear thermal expansion, vapor pressure, and electric resistivity. The first four volumes deal, respectively, with: I, elements (melting points above 1000° F.); II, alloys; III, ceramics; and IV, cermets, intermetallics, polymerics, and composites. Volume V contains the list of references from which the data were compiled and an author index.

The data are presented clearly, both in tabular and graphical form. Workers in missiles, aircraft and the nuclear sciences will find this handbook invaluable and easy to use; the sturdy binding will withstand hard usage.

AN INDEX OF PUBLISHED INFRA-RED SPECTRA, VOLS. I AND II, edited by Ministry of Aviation Technical Information and Library Services. 805 pages, 8½ × 13¼ in. London, Her Majesty's Stationery Office, 1960. Price, $18.40. American agents, British Information Services.

This "Beilstein" type of index enables a given spectrum to be traced in the literature. The first two volumes include most of the spectra published up to 1957, and subsequent volumes will follow. All the infra-red spectra are listed under a formula index; the state in which the sample was studied is also given, as are the wavelength range covered and an indication of the optical system employed. Nearly 10,000 references are covered. Practicing spectroscopists will find this a valuable source of information, and a real time-saver.

BLACK GOLD, by Arthur Beeby-Thompson. 533 pages, plates, 6½ × 9¼ in. New York, Doubleday & Co., Inc., 1961. Price, $5.95.

This is the autobiography of the Grand Old Man of oil, covering the exciting early days of this industry, beginning in 1898, up through the latest techniques. The reader is taken to Africa, Baku, the Caribbean, Mesopotamia, Peru, Egypt, America and Arabia, as the author recounts his adventures in pioneering in the oil industry.

MANAGEMENT MODELS AND INDUSTRIAL APPLICATIONS OF LINEAR PROGRAMMING, Volume 1, by A. Charnes and W. W. Cooper. 467 pages, diagrams, 7 × 10½ in. New York, John Wiley & Sons, Inc., 1961. Price, $12.50.

The contents of this new work are based on research conducted by the authors for almost ten years. It is directed mainly to persons who are interested in managerial applications of linear programming, and is intended to serve as a textbook as well as a reference.

Some of the general topics covered are the Simplex Method and Some Geometry in Linear Spaces, Data Interpretations by Direct and Dual Relations, Horizons and Surrogates in a Dynamic Model for Production Scheduling, and Degeneracy, Perturbation, and Complete Regularization.

QUANTUM THEORY: ELEMENTS, VOLUME 1, edited by D. R. Bates. 435 pages, diagrams, 6 × 9 in. New York, Academic Press Inc., 1961. Price, $10.00.

The first in a series of three volumes, this book is intended as an advanced text and reference on the fundamentals and applications of quantum theory. The reader should have a knowledge of classical mechanics, electromagnetic theory and atomic physics, and be familiar with the ordinary techniques of mathematical analysis.

In this first volume of *Quantum Theory*, non-relativistic wave mechanics and matrix mechanics are introduced and an extensive survey of the exactly soluble problems of the point and of the continuous spectrum is

presented. Developed in considerable detail are the approximate methods which are available for treating other stationary and time-dependent problems. Also treated fully is scattering theory, meeting the needs of both ionic and nuclear physicists.

OPERATIONAL ELECTRICITY, by Charles I. Hubert. 510 pages, diagrams, illustrations, 6 × 9 in. New York, John Wiley & Sons, Inc., 1961. Price, $8.50.

Intended as a text for students enrolled in a two-semester nonelectrical engineering course, this book deals with theory, characteristics, applications, and mode of operation of circuits and machines.

The author, who has had 18 years teaching experience in electrical engineering and is an associate professor of that course at the U. S. Merchant Marine Academy, has included in his latest book chapters on Circuit Elements, Balanced Polyphase System, Direct-Current Generators, Alternating-Current and Direct-Current Motors, Generator Voltage Regulators, Industrial Electronics and Instrumentation.

PROGRESS IN VERY HIGH PRESSURE RESEARCH, edited by F. P. Bundy, W. R. Hibbard, Jr., and H. M. Strong. 314 pages, diagrams, illustrations, 8¾ × 11¼ in. New York, John Wiley & Sons, Inc., 1961. Price, $12.00.

This book contains the proceedings of an International Conference held on June 13–14, 1960, at Bolton Landing, Lake George, New York. The conference was sponsored by the Materials Central, Wright Air Development Division of the U. S. Air Force and the General Electric Research Laboratory.

Since, in recent years, there has been a considerable increase in the use of very high pressure as a research method, it was felt that much progress could be achieved by bringing together some of the most active workers in the field for exchange of data, experience and philosophy. The editors hope that this book will help to stimulate and guide new work, and to encourage the continuation of current investigations.

The central topics of the conference were equipment and technique, structures of materials formed under high pressure, and behavior of matter under high pressure. Of particular interest in this book is a review of high pressure work in the USSR, presented by Professor L. F. Vereshchagin of Moscow.

TEMPERATURE MEASUREMENT IN ENGINEERING, VOL. II, by H. Dean Baker, N. H. Baker and E. A. Ryder. 465 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1961. Price, $13.50.

Volume II of this work completes the task begun in Volume I, published in 1953. Volume I dealt with thermocouples; Volume II is devoted to resistance thermometry and radiation pyrometry, as techniques in the measurement of temperature. Both are treated descriptively and analytically, as general techniques, in the first part of the book. The second, and larger, part contains a selected assortment of problems chosen from among the almost infinite variety available. In the applications portion of the book, chapters are devoted to such fields as surface temperature, moving bodies, liquids, gases, nuclear-reactor temperatures, and flame temperatures. In these, concrete examples of successful techniques are given; few worked-out examples are included.

THE COLLECTED WORKS OF IRVING LANGMUIR, compiled under the general editorship of Guy Suits. Twelve volumes, varying from approximately 250 to 500 pages, diagrams, 7 × 9¾ in. Oxford, Pergamon Press, 1960 & 1961. Price, $15.00 (£ 5), each volume.

This twelve-volume edition of the collected works of the first American industrial scientist to receive a Nobel Prize is being published under the general editorship of Guy Suits, one of Langmuir's associates in the General Electric Company's Research Laboratory. Langmuir's career was so varied and his scientific output so prolific that the editors have divided his contributions by subject matter, for inclusion in separate volumes. The first six, copyrighted in 1960 and 1961, are now available. They deal, respectively, with: 1, Low-Pressure Phenomena; 2, Heat Transfer—Incandescent Tungsten; 3, Thermionic Phenomena; 4, Electrical Discharge; 5, Plasma and Oscillations; and 6, Structure of Matter. Each volume contains, in addition

to Langmuir's own papers, an article analyzing his contributions to the particular field covered in that volume.

The editors and the publisher are to be congratulated for the care with which this important scientist's widely scattered contributions have been brought together for systematic presentation.

THE CELL AND THE ORGANISM, edited by J. A. Ramsay and V. B. Wigglesworth. 350 pages, diagrams, plates, 6 × 9¾ in. New York, Cambridge University Press, 1961. Price, $9.50.

This is a collection of essays on the application of experimental methods to different types of investigation in the zoological field. The subject matter covers a wide range, from the study of individual cells to the mechanics of motion in vertebrates. The contributions, written in honor of Sir James Gray, one of zoology's most eminent men, are original and, in some cases, speculative; they make a valuable addition to the literature on zoology and biology.

ENGINEERING MANAGEMENT AND ADMINISTRATION, by Val Cronstedt. 335 pages, diagrams, 6 × 8 in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $8.50.

Administrators who find themselves in charge of an engineering operation will find much of interest in this new book. Covering all phases of engineering administration except research, the book contains many helpful suggestions (on budget, proposals, test operations, patent law, records, etc.), designed to free the technical man from a military type of discipline.

THE CILIATED PROTOZOA, by John O. Corliss. Volume 7 of the International Series of Monographs on Pure and Applied Biology, Zoology Division, 310 pages, plates, 5¼ × 8¼ in. Oxford, Pergamon Press Ltd., 1961. Price, $12.00.

This monograph is written at the graduate level, but it should also be useful to teachers, post-graduate students and research workers in protozoology, parasitology and cell biology. The author proposes a new scheme of classification of the ciliated protozoa, based on new facts and revised interpretation of older data. Since this is the first work devoted solely to ciliated protozoa since 1935, such a reclassification seems called for, so that new families, genera and species may be fitted into the classification. A systematic index lists over 1400 scientific names of ciliate groups, and over 1600 literature references are cited.

OPTIMUM DESIGN OF MECHANICAL ELEMENTS, by Ray C. Johnson. 509 pages, diagrams, illustrations, 6 × 9 in. New York, John Wiley & Sons, Inc., 1961. Price, $11.50.

The author presents a method of mechanical design, a result of his many years of industrial and academic experience, which is explicit in nature and can be applied to innumerable original design problems. His method minimizes the "cut-and-try" practice, and takes into account the inherently unavoidable limitations which confront practicing mechanical design engineers.

The thirteen chapters include: Effects of Manufacturing Errors on Product Performance, Optimum Choice for Method of Analysis, Optimum Design of Torsion Shafts, and Optimum Design of Gears.

LINEAR SYSTEMS ANALYSIS, by Paul E. Pfeiffer. 538 pages, diagrams, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $12.50.

Intended to provide a theoretical core of fundamentals needed to understand a variety of systems and devices, this text stresses analytical tools and habits of analytical thought. The author does this through stressing: (1) physical analogies; (2) general rather than specific cases; and (3) self-contained mathematical treatment. As a text, the first seven chapters can be used as a one-semester introductory course for juniors; the entire book can be used on the senior level.

ELECTROMECHANICAL SYSTEM THEORY, by Herman E. Koenig and William A. Blackwell. 497 pages, diagrams, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $14.50.

The authors have prepared this text to present a new engineering systems discipline

which encompasses the wide variety of components found in the rapidly expanding control-systems field. In contrast to the notions of analogies used for many years, the application of linear graph theory is stressed as the core of the authors' proposed discipline. In this text, the subjects of rotating machinery, polyphase systems and servomechanisms have been integrated into the discipline, as well as those aspects of mechanics, hydraulics and electronic circuitry which are applicable in control systems. Emphasis is placed on precise formulation procedures inherent in linear graph theory. Prerequisites for study of the text are matrix algebra and Laplace transform techniques.

ADVANCES IN ELECTRONICS AND ELECTRON PHYSICS, VOL. XIV, edited by L. Marton and Claire Marton. 329 pages, diagrams, illustrations, 6 × 9 in. New York, Academic Press Inc., 1961. Price, $11.00.

The fourteenth volume of this well-known series contains the following six contributions: "The Electron as a Chemical Entity," by C. G. B. Garrett; "Problems of Photoconductivity," by P. Görlich; "Strong-Focusing Lenses," by Albert Septier; "Hydrogen Thyratrons," by Seymour Goldberg and Jerome Rothstein; "Cerenkov Radiation at Microwave Frequencies," by Herbert Lashinsky; and "High-Power Axial-Beam Tubes," by T. Moreno.

HANDBOOK OF SUPERSONIC AERODYNAMICS. SECTION 20. WIND TUNNEL INSTRUMENTATION AND OPERATION, by R. J. Volluz. 511 pages, diagrams, illustrations, 7½ × 10¼ in. Washington, Government Printing Office, 1961. Price, $3.00.

Section 20 of this *Handbook* compiled at Johns Hopkins is part of Volume 6, which covers design, construction and operation of the test facilities (free jets, shock tubes and wind tunnels) from which are obtained aerodynamic data. Section 20 deals with wind tunnel instrumentation and operation (design will be covered in Section 19). Three chapters cover in detail: various methods for measuring pressure, force, and temperature; the effects on such measurements produced by yaw, density, load, etc.; and errors associated with such measurements. Other chapters deal extensively with flow (Schlieren systems, interferometry, boundary-layer transitions, etc.), turbulence, tunnel blockage, model design, typical force measuring systems (reduction and plotting techniques), and pressure measuring systems (pressure transducers, manometers, etc.). This Section of the *Handbook* maintains the high standards set in the previously published sections.

PUBLICATIONS RECEIVED

FIELD GEOLOGY, by Frederic H. Lahee. Sixth edition, 926 pages, diagrams, illustrations, 5 × 7½ in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $10.75.

REACTOR HANDBOOK, VOL. II, FUEL REPROCESSING, edited by S. M. Stoller and R. B. Richards. Second edition, revised and enlarged, 665 pages, diagrams, illustrations, 7 × 10 in. New York, Interscience Publishers, Inc., 1961. Price, $21.40.

THE STORY OF ATOMIC THEORY AND ATOMIC ENERGY (Formerly titled THE ATOM STORY), by J. G. Feinberg. Unabridged republication of the first edition, with a new Foreword and an added chapter, 264 pages, diagrams, 5¼ × 8 in. New York, Dover Publications, 1961. Price, $1.45 (paper).

SPACE & TIME, by Emile Borel. Unabridged and unaltered republication of the first English translation, with a new Foreword by Banesh Hoffman, 234 pages, 5¼ × 8 in. New York, Dover Publications, 1961. Price, $1.45 (paper).

PIONEERS OF SCIENCE, by Sir Oliver Lodge. Unabridged and unaltered republication of the last corrected edition published in 1926. 404 pages, illustrations, 5¼ × 8 in. New York, Dover Publications, 1961. Price, $1.50 (paper).

WATER HAMMER IN HYDRAULICS AND WAVE SURGES IN ELECTRICITY, by Louis Bergeron. Translated from the 1949 French edition under the sponsorship of the Am. Soc. of Mech. Engineers, 293 pages, diagrams, inserts, 8½ × 11 in. New York, John Wiley & Sons, Inc., 1961. Price, $15.00.

FUNDAMENTALS OF PHYSICAL GEOGRAPHY, by Glenn T. Trewartha, Arthur H. Robinson and Edwin H. Hammond. 389 pages, diagrams, illustrations, maps, 7 × 9½ in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $6.95.

SOME ASPECTS OF LIFE IN FRESH WATER, by Edward J. Popham. Revised edition, 127 pages, diagrams, 4¾ × 7¼ in. Cambridge, Harvard University Press, 1961. Price, $2.25.

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Sapphire Whiskers for Silver.—Space scientists have increased the strength of pure silver five times by blending into it a small percentage of tiny sapphire "whiskers." At the same time they settled a long-standing question of whether the "whisker" technique can be used to produce ultra-strength metal materials.

Dr. Willard H. Sutton, of General Electric's Missile and Space Vehicle Department in Philadelphia, said similar techniques applied to high-strength metals can be expected to produce materials several times stronger at white heat than any high-strength alloy.

The purpose of the work, being done at the Department's Space Sciences Laboratory under a contract from the Bureau of Naval Weapons, is to help overcome one of the problems in space vehicle and advanced aircraft design—the heat barrier. There is a lack of structural materials that will maintain strength at the high temperatures created by reentry, hypersonic air speeds, and powerful jet and rocket engines.

The theory behind "whisker" reinforcement is:

Laboratory-grown whiskers, which consist of a single crystal of alumina (or sapphire), have shown extremely high strength—some can withstand tensions up to several million pounds per square inch. However, they have neither the rigidity nor the bulk to be machined or used as structural pieces or engine parts.

Currently known high-strength metals, including steels, can be machined into precision parts and are rigid, but at elevated temperatures (2000 deg. F. or more) they lose nearly all of their strength.

By embedding the whiskers inside the metal properly, the whiskers will carry nearly all of the load, thus permitting a metal to perform at temperatures far beyond its normal operating range. The principle is the same as that behind reinforced concrete.

The successful whisker-silver composite was made by causing molten silver to infiltrate a pack of whiskers in a vacuum.

The composite, containing 12½ per cent sapphire whiskers, held up under tension of 118,500 psi., which is five times the performance of the unreinforced metal, on a strength-to-weight basis.

The whiskers that went into the silver composite had an average strength of 750,000 psi. The over-all strength of the silver composite showed that the whiskers were not weakened by the combining process and that they were able effectively to bear the tensions.

These whiskers were grown from pure aluminum using a special heating process. To date the experiments have been concentrated on combining the whiskers with aluminum as well as silver. It is planned in the near future to develop composites using steel or other high-strength refractory metals as a matrix.

Computers for Evaluating Critical Fatigue and Alertness Levels.—Electronic computers can be invaluable tools in determining critical fatigue

levels and alertness of astronauts, airmen and motorists, a Radio Corporation of America scientist told the Fourth International Conference on Medical Electronics.

Dr. D. S. Himmelman, Administrator, Advanced Product Planning, RCA Electronic Data Processing Division, said research indicates an electronic "brain" can be used to excellent advantage in weeding out accident-prone individuals from among applicants for driver's licenses. At the same time, a computer can be employed in pinpointing the critical fatigue levels of astronauts and commercial airline pilots.

Dr. Himmelman said his findings were based on bio-medical research conducted in participation with Dr. Juan Negrin, Jr., of New York City, Dr. J. T. Chu of the University of Pennsylvania, and a group of volunteer subjects. Primary purpose of the study was to attain evaluation, through the use of electronic computers, of human capability to function under given stress conditions.

Initial results of the study showed that in addition to space, air and highway safety applications, military personnel on vital duty—such as radar watch—can be monitored for "indications of fatigue incompatible with continued efficient functioning."

In adjudging the risk quotient of would-be motorists, Dr. Himmelman said, the applicant for a driver's license would be tested behind the wheel of an automobile mock-up with a motion picture of varying highway conditions replacing the customary windshield. Electroencephalographic (EEG) signals emanating from the subject's brain would be transmitted to the electronic computer for comparison with previously-determined standards of acceptable reaction.

In the same manner, the RCA scientist said, EEG signals from space vehicles could be transmitted to a ground-based computer system to analyze the alertness of spacemen. Commercial air line pilots could be tested both before and after flights.

In the study by Dr. Himmelman and associates, the volunteer subjects were observed during periods of sleeplessness ranging up to 72 consecutive hours, interspersed with various forms of visual and other stimulation. The EEG signals from the individual's brain were converted to digital signals acceptable to an RCA 301 electronic data processing system. The computer's "memory" previously had been stored with factors having a bearing on the alertness readings.

The subjects ranged in age from 20 to 45, and they all underwent the tests in a controlled environment.

The wave forms monitored during the 72 hour period were related to a scale of scores ranging from zero to 20 reflecting alertness or capability to function in humans. On the basis of the EEG wave forms the computer pointed to the corresponding alertness score of the subject under test.

At present, Dr. Himmelman said, five points on the alertness scale are described by an initial prepared computer program, indicating a fatigue state equivalent to that induced by zero, 10, 20, 30, 40 and 50 hours of insomnia. Continued research and an increased sample of experimental subjects, will serve to improve the precision of the measurements, he said.

Fully Automatic Camera Fits Any Microscope.—The first fully automatic microphotographic camera that fits any microscope and has both detail and field-integrating exposure mechanisms has been developed by

Ernst Leitz GmbH of Wetzlar, West Germany, and introduced to medical researchers and industrial laboratory scientists in the U. S. by E. Leitz, Inc., American firm which distributes Leitz scientific instruments and the Leica camera line.

The new Orthomat micro-camera incorporates problem-solving features far more diverse in application than all available equipment. With a film magazine of 36 exposures, a researcher can rapidly take pictures of 36 different subjects and be assured of perfect exposures every time.

The extraordinary light-sensitive photocell is a photo multiplier tube, designed by R.C.A. This tube produces a current from the available light, charging a capacitor to a predetermined voltage. A gas discharge tube then starts the discharge, operating a relay; the electrical vibration-free shutter opens and closes, ending the exposure; and the film is transported by one frame. So automatic and versatile is the instrument that the researcher can operate the camera rapidly by foot pedal even as he watches through the microscope and makes adjustments with his hands.

The Leitz Orthomat is the first to achieve both built-in detail as well as integrating measurements, operated simply by two "push-buttons." An integrating measurement refers to an entire area. A detail measurement allows for correction of any particular detail within the microscopic field.

Another outstanding feature is that the film magazine can be changed at any time, place or interval. Exposures on color film can be changed at will to black and white work, and vice-versa, in two seconds. The self-contained 35-mm. camera has interchangeable cassettes to hold any type of film desired.

"It has always been a problem to determine the exact exposure necessary for a microscopic image," a company optical scientist said, "especially when conditions are bad, with dark field and tiny details. This instrument really takes care of an infinite variety of images from metallurgical samples to biological specimens."

The instrument is believed to be of profound significance to the country's pathologists, bacteriologists, medical technologists and others in the health field, as well as researchers in the steel, chemical, food processing, petroleum and a host of other industries. Virtually eliminating the chance of failure because of lack of particular knowledge of microphotography or other factors is considered an important achievement.

Many problem pictures are finally made easy. The range of exposure time is great. It includes those of photography of fluorescent specimens —½-hr. exposures and longer—and photography of living specimens, which have to be caught at 1/100th of a second.

Fuzed Fiber Optic Plates.—Fuzed fiber optic plates suitable for use as coupling elements in image tubes have recently been developed by the Warfare Vision Branch, U. S. Army Engineer Research and Development Laboratories, Fort Belvoir, Va.

A 1-in. diameter plate is made up of approximately 15,000,000 individual glass fibers fuzed into a solid piece with suitable optical insulation between fibers. Each fiber is only 0.00025 in. in diameter. These plates are able to transfer an image from one side of the piece through the glass channels to the other side, where the image is reformed by a series of shaded dots similar to a newspaper photo-

graph. The resultant image is high in resolution and contrast with negligible distortion. A program is now under way to place these elements in image tubes and other image intensification devices.

This work is part of a continuing program in the field of micro-optics. Future development will include fiber optic image inverters, translators and magnifiers for use in microminiature image intensification devices.

Silicon Photosensors.—Philco Corporation's Lansdale Division has announced a new line of silicon photosensors with high voltage capabilities making possible greater economies in the design of photo diode readout circuitry for data processing equipment. The new photosensors are capable of handling up to 20 volts of reverse breakdown voltage with under 10 μa of leakage current. They are designed as readout devices for use in conjunction with circuitry to convert light information into usable electrical signals such as pulses and sinusoidal waves. The photosensors' high voltage, low leakage current capabilities represent nearly two orders of magnitude of improvement over similar devices commercially available.

The improved characteristics offer a new economy to designers of data processing equipment using light detection as a readout method. For example, when used as light sensors for bias control in transistorized perforated tape or punched card readers, the improved voltage capabilities of these diodes permit the use of direct coupled circuitry. Devices now in use, having voltage handling capabilities of little more than 2 volts, are not adaptable to direct coupled circuitry, and therefore necessitate the use of transformers, incurrng an added cost. The improvement in voltage rating came about as a result of optimization of the diffusion process during production.

The new photosensors are hermetically sealed in glass envelopes of 0.080-in. diameter. Models are available with active area positioned parallel to the longitudinal axis of the diode, to provide a side-reading capability. Fast response time, low noise operation and high reliability charterize the devices. A second type has the active area positioned perpendicular to the longitudinal axis, and a lens for frontal exposure to the incident light eliminating the need for critical focusing.

Though designed primarily for punched card and tape read-out systems, other applications include—teletypewriter coding, production flaw testing, signal switching, photographic sensing and control, optical scanning and sound pickup.

Spectral response covers the range from visible to near infrared radiation, from 5000 angstroms to 11000 angstroms. Peak response is at 9500 angstroms.

"Plastinauts" for Radiation Research.—Performance testing of three man-size artificial figures designed for biological studies of radiation above the atmosphere began late this summer by the Air Force Systems Command.

The dummies are made of a rubber-based plastic covering a skeletal framework, and are instrumented to measure radiation doses in various parts of the body.

These "plastinauts" were constructed by Alderson Research Laboratories, Inc. of Long Island City, New York under a $35,000 contract for Systems Command's Air Force Special Weapons Center in Albuquerque, New Mexico.

Each dummy, which can be flown in any space vehicle designed to carry a man, weighs 162 lb. and is 5 ft. 9 in. tall. These dimensions are based on a 1950 anthropometric survey of more than 4,000 Air Force flying personnel.

The dummy's plastic covering absorbs radiation in a manner similar to human tissue. Space radiation will be measured by small ionization chambers inserted in various areas of the figures to correspond with organs of the human body. Such measurements are important to a clear understanding of biological effects of space radiation at different body depths.

High-strength fabric flying suits have been developed for the dummies by the David Clark Co. of Worcester Mass. The suit tightly encases each body segment and stabilizes the dummy in the position required.

The dummies are built to withstand the forces of missile launches: continuous loadings of 10 Gs and impacts of 25 Gs over a wide temperature range from 40° F. to 140° F.

Performance testing, conducted by the Biophysics Division, will include instrument calibration and structural strength studies.

Biophysicists will consolidate data received from the "plastinaut" flights with information collected from tissue-equivalent dose-rate-meters, small pound-and-a-half instruments which have been carried by research rockets to as high as 1,600 miles.

The dose-rate-meter uses lucite to simulate human tissue.

For the dummies, a tissue-simulating plastic based on an isocyanate rubber, duPont Adiprene-L, is used. Materials like lucite would be very difficult to adjust to correct radiological properties in an artificial figure and would not have the strength, toughness and resilience of the adiprene-base material. The adiprene-base plastic has an atomic makeup similar to human tissue, containing all the trace elements found in tissue.

Radiation-measuring instruments are inserted into pre-cast ducts. One duct is located in each arm and leg, one in the skull, and two each in the chest and abdomen areas of the figures. Electrical signals from the instruments are amplified electronically for telemetering to ground receiving stations.

Automation Applied to X-Ray Crystallography.—Research into the atomic structure of crystals will be significantly accelerated by a new X-ray diffraction technique which is almost entirely automatic. Developed by Dr. S. C. Abrahams of Bell Telephone Laboratories, it involves the use of a high-speed, general purpose, digital computer to generate a control tape which automatically operates a special type X-ray diffractometer. Data readings from the crystallographic experiment are automatically recorded on a punched tape which is then fed back into the computer. The computer correlates the information, corrects for experimental factors, and prints out crystallographic data in a form convenient for analysis by a crystallographer.

The new technique, called PEXRAD (Programmed Electronic X-Ray Automatic Diffractometer), was described in a paper by Dr. Abrahams at the American Crystallographic Association meeting in Boulder, Colorado. Dr. Abrahams said the equipment is in the final stages of construction and is expected to be in operation within a month.

Dr. Abrahams reported that with PEXRAD, it will be possible to obtain more than 17,000 readings a day; and while they are being taken auto-

matically, a crystallographer will be left free to evaluate collected data. With manual methods, the most an experienced crystallographer can hope to obtain is about 3000 readings a day, and then probably only for one or two days at a stretch since the measurements require his complete attention.

Dr. Abrahams said that the principles of PEXRAD are similar to an automatic neutron diffractometer method which he and Dr. E. Prince developed at Bell Laboratories in 1958. Single-crystal, automatic neutron diffractometers are currently being used in a number of research laboratories which have nuclear reactors.

He explained the PEXRAD method in detail: information such as crystal lattice constants, the wavelength of the X-ray beam, and instrumental constants are fed to the computer on punched cards, along with a compiling program. The computer generates a magnetic tape from which a punched paper tape is made. Fed into PEXRAD, it controls the motors which rotate the crystal and receiving counter into position, and then causes an X-ray beam of prescribed wavelength to irradiate the crystal for a precise length of time. The intensity of the beam which is diffracted by the crystal is measured and recorded on punched tape. The crystal is automatically rotated to the next position and the process is repeated.

The punched tape from PEXRAD containing the readings is converted to a magnetic tape and is fed back into the digital computer. The computer integrates the intensities, corrects for absorption, rotation, Lorentz and polarization factors, and prints out the corrected data from which the location of the individual atoms can be calculated.

Dr. Abrahams said that with PEXRAD, a crystal can be oriented with a hundredth-of-a-degree accuracy; therefore, it can be used to study almost any nonbiological crystal. The method could be extended to protein crystals, which need about one-thousandth-of-a-degree accuracy in crystal positioning.

Memory Storage Unit for Analog Computers.—Electronic Associates, Inc., Long Branch, N. J., has announced the first commercial sale of a general purpose analog computer equipped with a newly developed component that adds a high-speed memory storage capability. The memory component, called "Microstore," is transistorized and half the size of a carton of cigarettes. It has been under development and used experimentally for nearly a year. The commercial sale was made to a major chemical company for use in the study of chemical processes and associated control systems.

"Microstore," as an analog computer component, can store answers or values to problems for presentation later when the operator wants to compare new values to those previously computed. Two values at a time can be stored in one memory package for recall later. Analog computers, unlike digital, provide answers or values as electrical voltages that are displayed graphically as curves or oscilloscope displays.

Up to 10 "Microstore" components, each containing two storage units, can be plugged into an EAI PACE 231R general purpose analog computer without disturbing the use of existing summing amplifiers. According to the company, this design arrangement offers a distinct advantage, because the number of amplifiers that can be kept in operation determines the extent and

flexibility of the computer as a problem-solving instrument. The addition of the memory capability to the computer permits the solution of types of problems that were heretofore accomplished conveniently only by digital means.

The additional techniques made possible by "Microstore" can be applied to a multitude of problems in all industries where analog computation applies. The most immediate and probably best market is in the chemical and petrochemical fields for solving problems associated with various refinery processes. The new component can be used on existing EAI large-scale computers.

The company announcement also said that analog memory capability was first developed by EAI for an Air Force computer that was delivered in 1959. A second large-scale computer with memory capability was delivered earlier this year to the National Aeronautics and Space Administration research facility at Langley Field, Va., for use in simulating various aspects of Project Mercury, the "Man in Space" program. The company described this $1.5 million NASA computer as the largest general purpose analog computer ever delivered and said it was equipped with 45 memory storage units.

Electronic System for Monitoring Hospital Patients.—Tomorrow's hospital patients may have their temperature, pulse, and breathing under constant watch from one central point on each hospital floor by means of a simple "building block" system of tiny electronic measuring and radio transmitting devices described by scientists of the Radio Corporation of America at the Fourth International Conference on Medical Electronics.

Illustrating the concept was an experimental RCA radiocardiograph using a sensitive transmitter only slightly larger than a lump of sugar. Placed against the skin at the fingertips, on the arm, or elsewhere, the miniature unit detects heart action from the associated electrical changes in the body and transmits the information to a receiving antenna elsewhere in the room for display as a visible cardiogram on a viewing tube or an inked pattern on recording paper—thus eliminating the need for wire connections from the patient to the recorder.

The application of the principle to routine hospital measurements was described by Dr. Vladimir K. Zworykin, Honorary Vice President of RCA, and Fred L. Hatke, of the RCA Laboratories technical staff at Princeton, N. J. They said that similar miniaturized telemetering transmitters can now be developed to pick up and transmit a variety of such measurements, including temperature and rate of breathing as well as heart action. Built into tiny packages that can be conveniently worn by patients, they said, the devices can keep a constant check on these routine functions and transmit the information to an antenna installed in a room or ward. From the antenna, the data would be carried by wires to any central location for display to a floor nurse or doctor.

They pointed out that such measurements are now taken from time to time by the hospital staff as part of hospital routine to keep a watch on the general condition of each pateint. However, they added, "significant changes can occur and go completely undetected for some time during the period between successive measurements."

"With present methods of patient care, continuous monitoring of all

patients is impractical," they added. "However, the simple 'building block' technique of telemetry, with parts of the system added in successive steps, can be conveniently and economically used by all patients. And since all patients are in wards, semi-private, or private rooms, transmission is required over relatively short distances from the patient to an antenna."

Pointing out that such a system is now technically feasible, the RCA scientists predicted that its first application probably will be in surgical recovery rooms where continuous patient monitoring is essential.

Quartz Clock.—A new type of clock which is at least ten times as accurate as the best conventional chronometer has been developed at the General Electric Research Laboratory. The clock in its present form gains or loses only 5 seconds a year, and even better performance may be obtained if desired. The device is small and rugged, and operates for about four months on a single size D mercury battery (about the size of a standard flashlight cell). The clock is a laboratory device and no plans to produce it commercially have been announced.

The new clock, which was developed by Robert L. Watters, operates on an entirely different principle from that of an ordinary clock. Its timing standard is a quartz crystal, which produces very stable electrical vibrations at a high frequency. The quartz crystal has long been recognized as an almost ideal time standard, but its use in such a compact, practical chronometer became possible only with the recent development of the new solid-state electronic device, the tunnel diode. Previous clocks employing quartz crystals have required large amounts of electronic equipment to change the crystal's high frequency signal to the low frequency required to drive a clock motor. They also required bulky storage batteries to supply the large amounts of power needed.

Watters' clock, in contrast, uses only three tiny tunnel diodes to convert the 100,000 cycle-per-second crystal vibrations into a 50 cycle-per-second current to drive the clock motor. This is possible because of the unique ability of the tunnel diode to change a high frequency signal into one of much lower frequency, something that is relatively difficult to do using vacuum tubes or transistors. In fact Watters developed the new clock originally as a method of demonstrating the extreme stability of the tunnel diode in this application. Because of the small amount of power required by these electronic components, and a special low power clock motor produced by General Electric's Clock and Timer Department, the clock can operate for months on a small battery.

The tunnel diodes have proved to be so stable in operation that no inaccuracy in the clock can be ascribed to them. The only source of variation is the quartz crystal, which changes frequency very slightly with changes in temperature. In the clocks built so far, this inaccuracy amounts to only about five seconds per year over a wider temperature range than would be expected in normal conditions of use. By keeping the clock at a reasonably constant temperature, and perhaps by certain refinements in the electronic design, even greater accuracy can be obtained if desired.

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